Wireless network coding with partial overhearing information

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***Abstract*—We study an 1-hop broadcast channel with two receivers. Due to overhearing channels, the receivers have side information which can be leveraged by interflow network coding techniques to provide throughput increase. In this setup, we consider two different control mechanisms, the deterministic system, where the contents of the receivers' buffers are announced to the coding node via overhearing reports and the stochastic system, where the coding node makes stochastic control decisions based on statistics and the performance is improved via NACK messages. We study the minimal evacuation times for the two systems and obtain analytical expressions of the throughput region for the deterministic and the code-constrained region for the stochastic. We show that maximum performance is achieved by simple XOR policies. For equal transmission rates $r_1 = r_2$, the two regions are equal. If $r_1 \neq r_2$, we showcase the tradeoff between throughput and overhead.**

I. Introduction

Interflow network coding (coding together information from different flows) applied on multiple unicast flows has been shown to outpeform the classical routing schemes of the past. In 2006, Katti et al. demonstrated throughput benefits of interflow coding on real wireless devices using simple per-bit XOR operations. They designed and tested COPE, the first experimental evidence of practical *wireless network coding*, see [1], [2]. Apart from the classical two-way relay model (where 2 packets from two flows are delivered in three transmissions instead of four using XOR coding at the relay), COPE also proposed *opportunistic listening*; the nodes store opportunistically overheard packets in their buffers and then use them to decode future transmissions. This process increases throughput in cases of non-symmetric flows, as in Fig. 1.

Despite the warm reception from the research community, wireless network coding has not yet penetrated real applications. Although COPE and many important followup works (e.g. [3]–[6]) proposed distributed protocols which provide a packet level abstraction of coding operations to the above networking layers, there are still remaining issues which complicate the operation of such protocols and thus decelerate their actual deployment. One main concern, which is the focus of this work, is *how the coding node can be efficiently informed of the content of the decoding buffers of the receivers*.

The first approach, due to [1], is based on reporting overhearing events of all packets/receivers to all neighbors via an ACKing mechanism. Even though the proposed scheme is com-

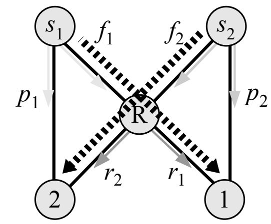

Fig. 1. Wireless network coding using overhearing channels - a two receiver case. Black lines represent links, dotted thick lines represent activated flows and gray arrows represent broadcast transmissions. Node *R* employs interflow network coding to improve the throughput achieved by the two flows.

pressing the information efficiently by sending coded reports in batches, the amount of circulated information is immense and there exist concerns about how timely these reports can be, especially when the scheme is used with very high packet rates. Another approach has been proposed by [1] (and studied further in [7]), where the coding node operates with statistics and feedback reports are used only when the receivers cannot decode a coded packet. This reduces the number of reports required but possibly the throughput performance as well, since the coding node makes decisions oblivious to the actual random overhearing events. An alternative approach is the I^{2}NC protocol, proposed in [8], which combines interflow with intraflow coding to reduce the complexity of acknowledgment messages at the expense of immediate decodability. In this work, we focus on the two first approaches and compare their performance in terms of maximum throughput and volume of feedback messages.

Consider the example of Fig. 1. Two non-symmetric flows are defined, $f_1 : s_1 \to 1$ and $f_2 : s_2 \to 2$. Both flows use the intermediate node *R* as a forwarder, which employs interflow network coding by XORing packets from the two flows. The receivers 1 and 2 utilize the overhearing erasure channels to obtain side information, i.e. packets destined to the other receiver. For example, receiver 1 receives packets destined to receiver 2, with probability p_2, whenever the source s_2 attempts to upload them to *R*. We focus on the downlink part which entails the complexity of the problem; node *R* must make coding and scheduling decisions in order to achieve some objectives, e.g. maximize throughput. In this context, we will call *deterministic* the system where node *R* learns the content of the decoding buffers of $1, 2$ via explicit reports that follow each overhearing event and *stochastic* the system where the decisions are made based on the probabilities of the overhearing channels and feedback reports following each unsuccessful attempt. This formulation is called 1-hop model.

A. Related work

Stability in networks with interflow network coding without overhearing is studied in [3] and [9]–[12]. Also, in [13], [14] the studies are extended to capture overhearing with reports, which corresponds to the deterministic system. Note that in these works, the code-constrained stability region is provided, i.e. the stability region under the assumption that XOR coding is used. The 1-hop model is also studied in [15] where the information theoretic capacity is given in the case of overhearing events provided as side information- a model equivalent to the deterministic system. With the exception of [14], all these works do not consider the stochastic system.

In [14], the stochastic system with feedback is studied under the assumption that receivers are not allowed to store coded packets and the code-constrained throughput region is provided in parametric form. The obtained throughput region is strictly smaller than that of the deterministic system. In this work, we extend [14] by allowing the storage of coded packets. We show, that if $r_1 = r_2$, then the stochastic system can achieve the same throughput as the deterministic one by the use of a simple XOR-based scheduling policy and feedback reports. Thus, the number of reports can be reduced significantly in this case without throughput losses.

Studies of the *broadcast channel with erasures*, i.e. see [16], relate to our work. In these studies, the problem is different since the side information for decoding is obtained from past erased transmissions; however, the techniques used are similar. In [17], the authors show that the capacity can be achieved by XOR coding for the case of 2-4 receivers. A different but related research topic is that of *index coding*; subsets of information bits are known to subsets of the receivers and we seek the transmission policy that minimizes the time to complete reception by all receivers, [18], [19]. Our work differs from index coding in the fact that the source has partial knowledge of what information each receiver has. Also, for the deterministic system, we extend the index coding problem to variable rates r_1, r_2.

Previous work has shown that in practical wireless networks, where the locations of the nodes are random, the vast majority of interflow coding opportunities involve a small number of nodes, [12], [20], [21]. This motivates the study of simple schemes with a small number of receivers, which can be solved efficiently. In this spirit, we provide optimal solutions that utilize simple XOR operations, require minimal information about system state, are oblivious to arrivals and can be embraced by resource limited wireless devices.

B. Contribution

We study the case of one node broadcasting coded transmissions and two receivers having side information. We allow the receivers to store any received or overheard packet (either native or coded) and use it in the future for decoding purposes.

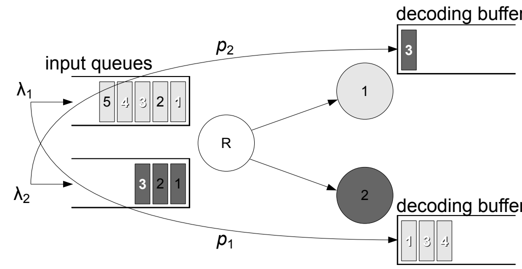

Fig. 2. The system under consideration; packets from two unicast flows arrive at the coding node *R* and are destined to two different receivers. Due to side overhearing channels, a copy of the arriving packet, destined to one receiver, also arrives at the other with a probability.

1) We give an outer bound for the throughput region of the deterministic system assuming general coding (including non-linear coding)-the equivalent information theoretic capacity region is shown in [15]. We show that this region can be achieved by simple XOR policies which operate without knowledge about arrivals.
2) For the stochastic system, we give in closed-form the code-constrained throughput region assuming the use of XOR coding. We propose a simple evacuation coding policy which achieves it. Interestingly, for the case of equal rates $r_1 = r_2$, this region is identical to the throughput region of the deterministic system.
3) We study performance tradeoffs between the two systems for a range of system parameters comparing the throughput efficiency and the feedback overhead.

II. System model

Consider a broadcast network with one transmitting node *R* (coding node) and two receivers $1, 2$. The time is slotted, where slot t occupies the time interval $[t, t+1)$. At the beginning of each slot, packets arrive at *R* with destination either receiver 1 or receiver 2. Within a slot, a number of packets are transmitted by *R*. We assume that all packets consist of L bits. The bits are i.i.d. with uniform distribution.

Arrivals. Packets arrive with the following property: whenever a packet destined to 1 (2) arrives at *R*, a copy of it arrives at 2 (1) with a probability p_1 (p_2). This probability corresponds to random overhearing events which are independent from one another. The packets arrive according to a stochastic arrival process with rate λ_1 and λ_2 correspondingly, see Fig. 2. We assume i.i.d. packet arrivals within each slot.

Storage. The coding node stores arriving packets in the input queues (hereinafter *queues*) while the receivers store packets useful for decoding in the decoding buffers (hereinafter *buffers*). As will be explained shortly, packets are separated into classes and are stored in queues corresponding to these classes. An obvious initial classification relates to the destination receiver, but we will expand the classification later. While the packets are saved in the queues in their *native form*, the buffers might contain packets in either native or coded form.

Transmissions. At the beginning of each slot t, *R* chooses a control (=a "type" of packets) c and transmits $r \in \mathbb{N}$ such packets. Controls are of two forms, c_i or $c_1 \oplus c_2$. Control

c_i refers to packets located in the queue c_i, where the index denotes the receiver i to which the packets are destined. Control $c_1 \oplus c_2$ refers to XOR coding of packets located to queues c_1 and c_2. We will relax the constraint of XOR coding in section IV-A.

Rates. Receiver i receives correctly all transmissions if r is less than or equal to r_i, $i = 1, 2$. Reception here corresponds to PHY layer operations, e.g. demodulation. For simplicity, we further constrain the actions of the coding node as follows. We let $r \in \{r_1, r_2\}$, and the coding node selects either r_i packets *directed* to receiver i or $\min\{r_1, r_2\}$ packets directed to both receivers. This way, the maximum number of packets is always transmitted in a given slot subject to correct reception at the involved receivers. Whenever the packets in a queue are less than the chosen r, *dummy* packets are used to fill in this number. We assume that the broadcast channel is erasure-free and we leave the study of erasures for future work.

XOR coding. The XOR coding refers to per bit modulo-2 additions of two packets x_1, x_2 denoted by $x_1 \oplus x_2$. The decoding is straight-forward if both the XORed packet and one of the native packets involved in the XOR combination are known to the receiver; applying a XOR addition on x_2 and $x_1 \oplus x_2$ provides x_1 for example. Whenever a received packet is not intended for the receiver (e.g. x_2 for receiver 1) or it cannot be decoded (e.g. $x_1 \oplus x_2$ when none of the native packets is available) it is stored for future use.

Departures. We assume that whenever a packet is obtained in native form by the intended receiver, the packet and all coded functions of it depart the system. In the following, we consider two different instances of the above problem.

A. Determistic system

Here we assume that the content of the buffers is announced to the coding node via a separate channel. The state of the system at time slot t is $S_{det}(t) = (k_1, k_2, n_1, n_2)$, where k_1 (k_2) is the number of packets destined to receiver 1 (2), and n_1 (n_2) is the number of packets destined 1 (2) and overheard by 2 (1). We call the latter, *good* packets due to their ability to be efficiently combined. The rest of the packets are called *bad* for consistency. Note, that since overhearing takes place only upon arrival, the categorization of good/bad does not change during the lifetime of a packet. Classifying the packets according to which receiver are destined, and whether they are good/bad, we use four queues to classify them upon arrival, named g_1, b_1, g_2, b_2. The control set is then defined as

$$\mathcal{C}_{det} \triangleq \{g_1, b_1, g_2, b_2, g_1 \oplus g_2\},$$

where, for example, control g_1 denotes the transmission of r_1 packets from queue g_1. The control $\{g_1 \oplus g_2\}$ is directed to both receivers (sent at rate $\min\{r_1, r_2\}$) and the controls $\{g_i, b_i\}$ are directed to receiver i (sent at rate r_i), $i = 1, 2$. Note, that we omit controls that apply XORs on bad packets. Although this is a constrained control set, we will show that optimal performance can be achieved using this set. A *policy* is a mapping from system state at the beginning of slot t to a control $c \in \mathcal{C}_{det}$, which corresponds to $r \in \{r_1, r_2\}$ transmissions, where r is determined by the chosen control. It is convenient to denote with $\mathcal{C}_{det}(t) \subseteq \mathcal{C}_{det}$ a subset of the control set with the property that the member controls correspond to non-empty queues. If for some t we have $\mathcal{C}_{det}(t) = \emptyset$, then clearly the system queues are empty.

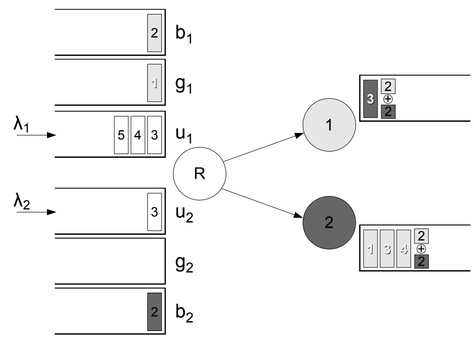

Fig. 3. A snapshot of the stochastic system. The packets are classified by the coding node as unknown (no knowledge of overhearing event), good (overheard) or bad (not overheard). We showcase the example of Fig. 2 after the control $u_1 \oplus u_2$ is used with $r = 2$.

B. Stochastic system

The coding node estimates the overhearing events based on probabilities p_1, p_2 and uses feedback to improve the estimation. We assume a feedback mechanism that announces to the coding node the inability of a receiver to decode a coded packet using NACK messages. For a transmitted packet $x_1 \oplus x_2$, the mechanism is used by the coding node as follows; if no NACKs are received, then both packets depart the system as described above. If x_1 is NACKed but x_2 not, then the latter departs the system and the coding node obtains the information that receiver 2 has x_1 and receiver 1 has $x_1 \oplus x_2$. In this case, x_1 is put in the g_1 queue, while the coded packet is not stored since it is a function of the departed packet. The symmetric case where x_2 is NACKed but x_1 not, is obtained by exchanging 1 and 2. Finally, if both packets are NACKed, then both receivers have $x_1 \oplus x_2$ and both packets are stored in the corresponding queues b_1, b_2. It should be noted that all packets in bad queues are associated with the knowledge that a XOR function is stored in the buffers. See Fig. 3, 2 for an example of operation.

Upon arrival, the packets are classified as *unknown* since the coding node only possesses stochastic knowledge about the corresponding overhearing events. For this reason, queue u_i for unknown packets is introduced and all arrivals enter the coding node at these queues, see Fig. 3. The packets may leave this queue when they depart the system or if moved to another queue according to the above-described mechanism.

The system state is $S_{det}(t) = (k_1, k_2, n_1, n_2, m_1, m_2)$, where k_i is the total number of packets of flow i in the queues, n_i the number of packets in g_i and m_i the number of packets in b_i. We define the control set as:

$$\mathcal{C}_{sto} \triangleq \{g_1, b_1, u_1, g_2, b_2, u_2, g_1 \oplus g_2, g_1 \oplus u_2, u_1 \oplus g_2, u_1 \oplus u_2\}.$$

The set is again constrained to exclude XOR controls involving packets from the bad queues. This happens without loss of optimality, though a proof is omitted here due to lack of space. Controls $\{g_i, u_i\}$ are directed to receiver i (as before), while the rest controls are directed to both receivers. The policies and set $\mathcal{C}_{sto}(t)$ are defined as in the deterministic case. We will refer to controls $\{g_1, b_1, \dots\}$ as *single* controls and to $\{g_1 \oplus g_2, \dots\}$ as *XOR* controls, denoting the corresponding sets with $\mathcal{C}^{\mathrm{s}}$, $\mathcal{C}^{\mathrm{x}}$.

III. Stability considerations

Consider the set of queues at the coding node, denoted Q. Denote the sum of backlogs of queues in Q under policy σ at the end of time slot t as $X_i^\sigma(t)$. As in [22], we say that the system is stable if

$$\lim_{q\to\infty} \limsup_{t\to\infty} \Pr\left(X_i^\sigma(t) > q\right) = 0.$$

Note that the definition of stability does not include the buffers. Due to the definition of departures, though, stability of queues implies stability for the buffers.

Consider the set of all vectors $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ for which the system is stable under policy σ; the closure of this set denoted by Λ^σ is called the *stability region* of the policy σ. The region $\Lambda \triangleq \cup_\sigma \Lambda^\sigma$ characterizes the system and is called the *throughput region*. In case we constrain the allowable set of codes (e.g. to XOR only) we will refer to the corresponding region as the *code-constraint throughput region*, see [19].

We expect the code-constraint region of the stochastic system to be a subset of the throughput region of the deterministic due to the partial information available at the coding node and the restriction to XORing.

A. Evacuation time and stability

In order to study the stability of the described model, we consider a different operation of the system, which is based on evacuating system snapshots. We assume a snapshot with k_1 packets destined to receiver 1 and k_2 destined to receiver 2, where the packets have arrived following the rules explained above regarding overhearing. The number of overheard packets is random; denote by $n_1 \triangleq N_1(\omega)$ the number of packets destined to receiver 1 and overheard by receiver 2 and $n_2 \triangleq N_2(\omega)$ similarly. Then, the system operates as it would normally do with the difference that no extra arrivals are introduced in the system. An admissible *evacuation policy* π is a sequence of eligible control actions at the end of which all packets have departed from the system. Note, that each evacuation policy can be mapped to an epoch-based policy $\sigma(\pi)$, which is admissible in the system with arrivals and evacuates all packets present in the system at the beginning of each epoch using π, see [22]. Next, we follow the steps of [22].

Let Π be the set of all evacuation policies. We denote with $T^\pi(k_1, k_2, n_1, n_2)$ the *evacuation time* of policy $\pi \in \Pi$, which is the minimum number of slots required to empty the system queues under policy π. We denote with $\overline{T}^\pi(k_1, k_2) \triangleq \mathbb{E}[T^\pi(k_1, k_2, N_1, N_2)]$ the average evacuation time of this policy and with $\overline{T}^\star(k_1, k_2) \triangleq \inf_{\pi\in\Pi}\{\overline{T}^\pi(k_1, k_2)\}$ the minimum average evacuation time over all the policies.

Lemma 1 [Subadditivity and linear growth]: *The function $\overline{T}^\star(k_1, k_2)$ is subadditive, is upper bounded by a linear function and the following limit exists*

$$\hat{T}(\lambda_1, \lambda_2) = \lim_{t\to\infty} \frac{\overline{T}^\star(\lceil t\lambda_1 \rceil, \lceil t\lambda_2 \rceil)}{t}.$$

Proof: In [22], Lemma 1 is shown under a general class of policies, provided that these policies have certain Features and under some Assumptions on System operation, all of which hold trivially in our problem. ■

Proposition 2 [Throughput region via evacuation times from [22]]: *The throughput region of the system is the set of rates $(\lambda_1, \lambda_2) \geq (0, 0)$ satisfying*

$$\hat{T}(\lambda_1, \lambda_2) \leq 1.$$

IV. Analysis of the Deterministic system

For the purposes of this section, we will allow arbitrary coding functions (including non-linear coding) on any subset of packets, relaxing the restriction of XORing only two packets from different flows. This way, we provide a lower bound on the minimal evacuation time $\overline{T}^\star(k_1, k_2)$ and correspondingly, its linear growth. Then, we show that simple XOR-based online policies, which operate agnostically to arrival rates, can be used to evacuate the system with the same growth. This in turn establishes the throughput region for the deterministic system, which is given in a closed-form expression.

A. Lower bound on evacuation time under general coding

The development of the lower bound is based on a preliminary result which we present next. Let $\mathcal{X}, \mathcal{Y}, \mathcal{M}_1, \mathcal{M}_2$ be finite sets. Consider sequences $X_l \in \mathcal{X}$, $l = 1, \dots, k_1$ and $Y_l \in \mathcal{Y}$, $l = 1, \dots, k_2$. Denote $A^K \triangleq (A_1, \dots, A_k)$. We also consider two coding functions $\Phi_1 : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1$, $\Phi_2 : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_2$ and two decoding functions $g_1 : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{Y}^{n_2} \to \mathcal{X}^{k_1}$ and $g_2 : \mathcal{M}_1 \times \mathcal{X}^{n_1} \to \mathcal{Y}^{k_2}$, where $0 \leq n_i \leq k_i,\ i = 1, 2$. We impose error-free decoding:

Condition 1 [Decoding]: *For any (X^{k_1}, Y^{k_2})*

1) $g_1\left(\Phi_1(X^{k_1}, Y^{k_2}), \Phi_2(X^{k_1}, Y^{k_2}), Y^{n_2}\right) = X^{k_1}$.
2) $g_2\left(\Phi_1(X^{k_1}, Y^{k_2}), X^{n_1}\right) = Y^{k_2}$.

Fix Y^{n_2} and define the mapping $\Psi : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2 - n_2} \to \mathcal{M}_1 \times \mathcal{M}_2$, where

$$\Psi\left(X^{k_1}, Z^{k_2-n_2}\right) = \left(\Psi_1\left(X^{k_1}, Z^{k_2-n_2}\right), \Psi_2\left(X^{k_1}, Z^{k_2-n_2}\right)\right),$$

$$\Psi_l\left(X^{k_1}, Z^{k_2-n_2}\right) = \Phi_l\left(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}\right),\ l = 1, 2,$$

$Y^{n_2}||Z^{k_2-n_2}$ is the concatenation of sequences Y^{n_2},$Z^{k_2-n_2}$.

Similarly fix X^{n_1} and define the mapping $\Theta : \mathcal{X}^{k_1-n_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1 \times \mathcal{M}_2$, with

$$\Theta_l\left(Z^{k_1-n_1}, Y^{k_2}\right) = \Phi_l\left(X^{n_1}||Z^{k_1-n_1}, Y^{k_2}\right),\ l = 1, 2.$$

We also define the mapping $\Phi : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1 \times \mathcal{M}_2$ as,

$$\Phi\left(X^{k_1}, Y^{k_2}\right) = \left(\Phi_1\left(X^{k_1}, Y^{k_2}\right), \Phi_2\left(X^{k_1}, Y^{k_2}\right)\right).$$

Lemma 3: *Under condition 1, for any fixed Y^{n_2} (fixed X^{n_1}) the mapping Ψ (Θ) is injective. Hence it holds,*

$$|\mathcal{R}(\Psi)| = |\mathcal{X}|^{k_1}|\mathcal{Y}|^{k_2-n_2}, \quad |\mathcal{R}(\Theta)| = |\mathcal{X}|^{k_1-n_1}|\mathcal{Y}|^{k_2}$$

where $\mathcal{R}(\Phi)$ denotes the range of a mapping Φ. Moreover, for any fixed X^{k_1}, the mapping $\tilde{\Phi}_1 : |\mathcal{Y}|^{k_2} \to \mathcal{M}_1$ defined by $\tilde{\Phi}_1\left(Y^{k_2}\right) = \Phi_1\left(X^{k_1}, Y^{k_2}\right)$ is injective, hence

$$\left|\mathcal{R}(\tilde{\Phi}_1)\right| = |\mathcal{Y}|^{k_2} \tag{1}$$

Proof: To show that Ψ is injective, it suffices to show that if
$$\Psi_l\left(X^{k_1}, Z^{k_2-n_2}\right) = \Psi_l\left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), l = 1, 2,$$
then $X^{k_1} = \hat{X}^{k_1}$, and $Z^{k_2-n_2} = \hat{Z}^{k_2-n_2}$. We write

$$\begin{aligned} X^{k_1} &= g_1\left(\Phi_1(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), \Phi_2(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), Y^{n_2}\right) \\ &= g_1\left(\Psi_1(X^{k_1}, Z^{k_2-n_2}), \Psi_2(X^{k_1}, Z^{k_2-n_2}), Y^{n_2}\right) \\ &= g_1\left(\Psi_1\left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), \Psi_2\left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), Y^{n_2}\right) \\ &= g_1\left(\Phi_1(\hat{X}^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}), \Phi_2(\hat{X}^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}), Y^{n_2}\right) \\ &= \hat{X}^{k_1} \end{aligned}$$

$$\begin{aligned} Y^{n_2}||Z^{k_2-n_2} &= g_2\left(\Phi_1(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), X^{n_1}\right) \\ &= g_2\left(\Psi_1(X^{k_1}, Z^{k_2-n_2}), X^{n_1}\right) \\ &= g_2\left(\Psi_1\left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\ &= g_2\left(\Psi_1\left(X^{k_1}, \hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\ &= g_2\left(\Phi_1\left(X^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\ &= Y^{n_2}||\hat{Z}^{k_2-n_2} \end{aligned}$$

Hence, $Z^{k_2-n_2} = \hat{Z}^{k_2-n_2}$. To prove Θ is injective, we argue similarly starting from the decoding function g_2. Finally, (1) follows by the fact that for any X^{k_1}, if $\Phi_1(X^{k_1}, Y^{k_2}) = \Phi_1(X^{k_1}, \hat{Y}^{k_2})$, then $Y^{k_2} = \hat{Y}^{k_2}$, which again follows from

$$g_2\left(\Phi_1(X^{k_1}, Y^{k_2}), X^{n_1}\right) = g_2\left(\Phi_1(X^{k_1}, \hat{Y}^{k_2}), X^{n_1}\right).$$
■

COROLLARY 4: *Assume that X^{k_1} and Y^{k_2} consist of independent identically distributed random variables and are independent of each other. Then the mappings Φ_1, Φ_2 are random variables and it holds,*

$$H\left(\Phi_1\right) \geq k_2 H\left(Y\right) \tag{2}$$

$$\begin{aligned} H\left(\Phi\right) \geq \max\{&k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right), \\ &\left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)\} \end{aligned} \tag{3}$$

Proof: Since for fixed X^{k_1} the mapping $\tilde{\Phi}_1$ is injective

$$\begin{aligned} H\left(\Phi_1 \left| X^{k_1} = x^{k_1}\right.\right) &= H\left(\tilde{\Phi}_1 \left| X^{k_1} = x^{k_1}\right.\right) \\ &= H\left(Y^{k_2}\right) = k_2 H(Y). \end{aligned}$$

Hence, $H\left(\Phi_1\right) \geq H\left(\Phi_1 \left| X^{k_1}\right.\right) = k_2 H\left(Y\right)$. Also

$$H\left(\Phi\right) \geq H\left(\Phi \left| Y^{n_2}\right.\right) = k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right)$$
$$H\left(\Phi\right) \geq H\left(\Phi \left| X^{n_1}\right.\right) = \left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)$$

are derived in a similar fashion. ■

The interpretation of this formulation in the context of the current paper is the following: $\mathcal{X}$ and $\mathcal{Y}$ are all possible L-bit sequences that can be contained in a packet, $|\mathcal{X}| = |\mathcal{Y}| = 2^L$. The sequence X^{k_1} represents the k_1 packets at the transmitter that are destined to receiver 1, while the subsequence X^{n_1} represents the packets at the transmitter that are destined for receiver 1 and have been overheard by receiver 2. The interpretation of sequence Y^{k_2} and its subsequence Y^{n_2} is similar. Since bit sequences are assumed i.i.d with uniform distribution, we have $H\left(X\right) = H\left(Y\right) = L$.

For the rest of the discussion we assume that $r_1 \geq r_2$, hence receiver 1 observes all slots, while receiver 2 observes only slots at which packets are transmitted at rate r_2. The set $\mathcal{R}(\Phi_1)$ represents the values of the mapping which must be known to receiver 2 so that together with X^{n_1} successful decoding is effected at this receiver. Therefore, the values of $\mathcal{R}(\Phi_1)$ must be transmitted during slots at which the rate is r_2. We denote by ξ_2 the (random) number of packets transmitted during these slots, and by $\overline{\xi}_2$ its average value. Hence, the average number of bits transmitted in slots with rate r_2 is $\overline{\xi}_2 L$. Similarly, for the set $\mathcal{R}(\Phi)$ and receiver 1. We denote by ξ the (random) number of slots used in the transmission of all the packets to both receivers, and by $\overline{\xi}$ its average value.

In order for the receivers to obtain the values of the sets $\mathcal{R}(\Phi_1)$, $\mathcal{R}(\Phi)$ these values must be source-coded and transferred through the channel using packets of L bits. We use uniquely decodable codes and hence the average number of bits that need to be transmitted is bounded from below as follows.

To transfer the values of $\mathcal{R}(\Phi_1)$, using (2)

$$\bar{\xi}_2 L \geq H\left(\Phi_1\right) \geq k_2 H(Y) = k_2 L \tag{4}$$

Similarly, to transfer the values of $\mathcal{R}(\Phi)$, using (3)

$$\begin{aligned} \bar{\xi} L &\geq H\left(\Phi\right) \geq \max\left\{H(\Psi), H(\Theta)\right\}. \\ &\geq \max\{k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right), \\ &\qquad \left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)\} \\ &= L \max\left\{k_1 + \left(k_2 - n_2\right), \left(k_1 - n_1\right) + k_2\right\} \\ &= L\left(k_1 + k_2 - \min\left\{n_1, n_2\right\}\right) \end{aligned} \tag{5}$$

THEOREM 5 [LOWER BOUND WITH ARBITRARY CODING]: *The deterministic system satisfies under any $\pi \in \Pi$:*

$$\overline{T}^{\pi}(k_1, k_2) \geq T^{bdet}(k_1, k_2),$$

where $T^{bdet}(k_1, k_2) \triangleq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[\min\{N_1, N_2\}]}{\max\{r_1, r_2\}}$.

Proof: Assume without loss of generality $r_1 \geq r_2$. Also, let $\xi_1 \triangleq \xi - \xi_2$ be the number of packets transmitted during slots where rate r_1 is used, so that only receiver 1 observes them. For any policy π we have

$$T^{\pi}(k_1, k_2, n_1, n_2) \geq \left\lceil \frac{\xi_2}{r_2} \right\rceil + \left\lceil \frac{\xi_1}{r_1} \right\rceil \geq \frac{\xi_2}{r_2} + \frac{\xi - \xi_2}{r_1}.$$

Taking into account (4), (5) we then have

$$\begin{aligned} &\bar{T}^{\pi}(k_1, k_2, n_1, n_2) \geq \frac{\overline{\xi} - \overline{\xi}_2}{r_1} + \frac{\overline{\xi}_2}{r_2} \\ &\geq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[\min\{N_1, N_2\}]}{r_1} + \left(\bar{\xi}_s - k_2\right)\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \\ &\geq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[\min\{N_1, N_2\}]}{r_1} \end{aligned}$$

The result follows by using the same methodology for the case of $r_2 > r_1$. ■

B. A class of simple XOR-based policies

DEFINITION 1 [CLASS Π_{DET}]: *At slot t the control is chosen according to the following two steps:*

1) *If $\{g_1 \oplus g_2\} \in \mathcal{C}_{det}(t)$, choose control $\{g_1 \oplus g_2\}$.*
2) *Else, choose any single control (each policy in the class defines a different order).*

When $\mathcal{C}_{det}(t) = \emptyset$, stop.

Let $\overline{r} \triangleq r_1$ if $n_1 \geq n_2$ and $\overline{r} \triangleq r_2$, otherwise. Notice, that since the policies in Π_{det} do not depend on the values of the bits in the packets, their evacuation times are deterministic. By enumerating the two above steps, we have

$$T^\pi(k_1, k_2, n_1, n_2) \leq \left\lceil \frac{\min\{n_1, n_2\}}{\min\{r_1, r_2\}} \right\rceil + $$
$$+ \left\lceil \frac{\max\{n_1, n_2\} - \min\{n_1, n_2\}}{\overline{r}} \right\rceil + \left\lceil \frac{k_1 - n_1}{r_1} \right\rceil + \left\lceil \frac{k_2 - n_2}{r_2} \right\rceil$$
$$\leq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\min\{n_1, n_2\}}{\max\{r_1, r_2\}} + 4, \quad \text{for all } \pi \in \Pi_{\text{det}}. \quad (6)$$

THEOREM 6 [THROUGHPUT REGION]: *The throughput region of the deterministic system is the area defined by $(\lambda_1, \lambda_2) \geq (0, 0)$, and the following inequality:*

$$\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{p_{12}\lambda_1, p_{12}\lambda_2\}}{\max\{r_1, r_2\}} \leq 1. \quad (7)$$

Proof: Consider $k_i = \lceil \lambda_i t \rceil$, $i = 1, 2$ packets to be evacuated and note that the number of good packets per flow are binomial random variables, denoted by $N_1(k_1), N_2(k_2)$ correspondingly. By the fact that status of arriving packets (good, bad) is an i.i.d. process, we have,

$$\lim_{t \to \infty} \mathbb{E}\left[\frac{N_i(\lceil \lambda_i t \rceil)}{t}\right] = p_i \lambda_i, \quad (8)$$

and also, by the strong law of large numbers,

$$\lim_{t \to \infty} N_i(\lceil \lambda_i t \rceil, \omega)/t = p_i \lambda_i \quad \text{w.p.1.} \quad (9)$$

Recall that $\overline{T}^\star(k_1, k_2)$ is the minimum average evacuation time over all policies, hence smaller than $\overline{T}^\pi$. By Theorem 5

$$\mathbb{E}\big[T^{bdet}(k_1, k_2, N_1, N_2)\big] \leq \overline{T}^*(k_1, k_2) \leq \mathbb{E}[T^\pi(k_1, k_2, N_1, N_2)]. \quad (10)$$

We calculate the limit of the upper bound of $\overline{T}^\pi(k_1, k_2)$ using the the RHS of (6)

$$\lim_{t \to \infty} \mathbb{E}\left[\frac{\frac{k_1 t}{r_1} + \frac{k_1 t}{r_1} - \frac{\min\{N_1(k_1 t, \omega), N_2(k_2 t, \omega)\}}{\max\{r_1, r_2\}} + 4}{t}\right] =$$
$$= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \lim_{t \to \infty} \mathbb{E}\left[\frac{\min\{N_1(k_1 t, \omega), N_2(k_2 t, \omega)\}}{t \max\{r_1, r_2\}}\right]$$
$$= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\min\{p_1 k_1, p_2 k_2\}}{\max\{r_1, r_2\}}, \quad \text{w.p.1},$$

where in the last step, we exchange the order of limit expectation and min function due to uniform integrability which follows from convergence in expectation (8) and almost everywhere convergence (9) of the involved sequences, see [23] Th. 16.14. We can repeat the limit derivation for the case of T^{bdet} and derive the same limit, hence from (10) we conclude

$$\hat{T}(\lambda_1, \lambda_2) = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{p_1\lambda_1, p_2\lambda_2\}}{\max\{r_1, r_2\}}$$

and the result follows by invoking Proposition 2. ■

V. ANALYSIS OF THE STOCHASTIC SYSTEM

In this section, we study a set of evacuation policies Π for the stochastic system, constrained to the use of XORs (i.e. general coding is not considered) and derive the corresponding code-constrained throughput region in closed-form.

A. Treating packets in queues b_1, b_2

We focus on a special control sequence. Following a control $u_1 \oplus u_2$, and two NACK messages from the receivers, the corresponding transmitted packets x_1, x_2 are characterized as bad and put in the corresponding bad queues. Assume, that in a succeeding time slot, one of the two packets is transmitted using a single control, say b_1, directed to both receivers (i.e. at rate $r = \min\{r_1, r_2\}$). Evidently receiver 1 will obtain x_1, which departs the system. Since receiver 2 has previously obtained the coded packet $x_1 \oplus x_2$ from the NACKed broadcast transmission, receiver 2 can combine it with x_1 and obtain x_2. In a total of two transmissions, both bad packets are obtained.

Due to the control set $\mathcal{C}_{\text{sto}}$, the packets in the bad queues can only be evacuated by single controls. Since, for each bad packet in the queue b_1 there is a bad packet in queue b_2 (the one with which it was coded), for efficiency reasons we will assume that they are always directed to both receivers. Finally, since the evolution of the system state is not affected, we will constrain the set of evacuation policies to those that choose controls b_1, b_2 last.

B. Lower bound on the code-constrained growth rate

Let $(f, s) = (1, 2)$ if $r_1 \geq r_2$ and $(f, s) = (2, 1)$ otherwise, where *f*=fast and *s*=slow. Also, let $(.)^+ \triangleq \max\{., 0\}$ and

$$B^{req} \triangleq \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min(\lambda_1 p_1, \lambda_2 p_2)}{p_f}\left[\frac{1}{r_f} - \frac{1 - p_f}{r_s}\right]^+.$$

THEOREM 7 [LOWER BOUND ON THE GROWTH RATE]: *For the stochastic system, constrained to the use of XOR coding, it holds*

$$\liminf_{t \to \infty} \frac{\overline{T}^\pi(\lceil t\lambda_1 \rceil, \lceil t\lambda_2 \rceil)}{t} \geq B^{req}, \text{ for all } \pi \in \Pi. \quad (11)$$

The proof is in the Appendix.

C. An optimal evacuation policy

DEFINITION 2 [POLICY π^*]: *Policy $\pi^* \in \Pi$ operates as follows. If $1 - p_f > \frac{\min(r_1, r_2)}{\max(r_1, r_2)}$ is true, controls from $\mathcal{C}^s$ are chosen in arbitrary order. Else, at slot t*

- *If $\{u_1 \oplus g_2\} \in \mathcal{C}_{sto}(t)$ or $\{u_2 \oplus g_1\} \in \mathcal{C}_{sto}(t)$, then select the corresponding control*
- *elseif $\{u_1 \oplus u_2\} \in \mathcal{C}_{sto}(t)$ select this control*
- *else select any control from the set $\mathcal{C}^s$. During this step, controls b_1 and b_2 are used in the way explained in subsection V-A.*

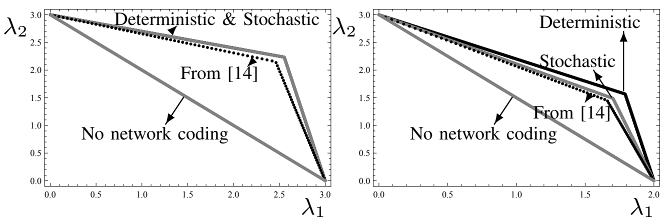

Fig. 4. Throughput regions of no network coding and deterministic system and code-constrained regions of the stochastic system with or without storing XORs (from [14]). Parameters: $p_1 = 0.7, p_2 = 0.8, r_2 = 3$ and $r_1 = 3$ (left), $r_1 = 2$ (right).

When $\mathcal{C}_{sto}(t) = \emptyset$ stop.

THEOREM 8 [ASYMPTOTIC OPTIMALITY OF POLICY π^*]: *For the stochastic system operating under policy π^* we have*

$$\limsup_{t\to\infty} \frac{\overline{T}^{\pi^*}(\lceil t\lambda_1\rceil, \lceil t\lambda_2\rceil)}{t} \leq B^{req}. \tag{12}$$

The proof of Theorem 8 is in the Appendix. Combining (11) and (12), we conclude $\hat{T}^\star(\lambda_1, \lambda_2) = B^{req}$.

COROLLARY 9: *The code-constrained region of the stochastic system is the set of rates $(\lambda_1, \lambda_2) \geq (0,0)$ satisfying*

$$\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{\lambda_1 p_1, \lambda_2 p_2\}}{p_f}\left[\frac{1}{r_f} - \frac{1-p_f}{r_s}\right]^+ \leq 1, \tag{13}$$

where $r_s = \min\{r_1, r_2\}$ and $r_f = \max\{r_1, r_2\}$. Whenever the term in the brackets is negative, network coding is not beneficial, and the maximum throughput is achieved without coding. If $r_1 = r_2$, the terms cancel out and (13) equals (7), therefore the code-constrained region of the stochastic system and the throughput region of the deterministic are equal. In Fig. 4 we plot the regions for two different settings.

VI. NUMERICAL COMPARISON

In this section, we study the throughput-overhead tradeoff between the deterministic and the stochastic systems.

For the deterministic system, assuming $\mathcal{N}_1, \mathcal{N}_2$ represent the set of neighboring nodes of sources $1, 2$, the average rate of overhearing reports $W^{det}(\lambda_1, \lambda_2)$ is calculated as

$$W^{det}(\lambda_1, \lambda_2) = \lambda_1 \sum_{i\in\mathcal{N}_1 - R} p_{1,i} + \lambda_2 \sum_{i\in\mathcal{N}_2 - R} p_{2,i},$$

where $1 - p_{1,i}$ is defined to be the erasure probability of the link from source 1 to neighbor i and $1 - p_{2,i}$ likewise. For the stochastic system, note that one NACK message is associated with each bad packet, since good packets are transmitted without feedback messages. Thus, using policy π^*, the corresponding rate is

$$W^{sto}(\lambda_1, \lambda_2) = q_1^{\text{XOR}}(1-p_1)\lambda_1 + q_2^{\text{XOR}}(1-p_2)\lambda_2,$$

where q_i^{XOR} is the fraction of bad packets of flow i that are transmitted using XOR controls. If $1 - p_f \leq \frac{\min(r_1,r_2)}{\max(r_1,r_2)}$, no coding is performed and thus $q_1^{\text{XOR}} = q_2^{\text{XOR}} = 0$. Else, we can find an upper bound, when (λ_1, λ_2) lies on the boundary of the throughput region.

$$\overline{q}_1^{\text{XOR}} = \begin{cases} 1 & \text{if } \lambda_1 p_1 \leq \lambda_2 p_2 \\ \frac{\lambda_2 p_2}{\lambda_1 p_1} & \text{otherwise.} \end{cases}$$

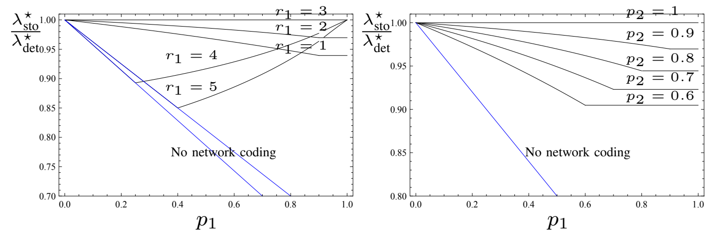

Fig. 5. **Throughput efficiency** varying p_1. Parametric plots vs r_1 (left) and vs p_2 (right). Default parameters: $\lambda_1 = \lambda_2$, $p_2 = 0.9$, $r_1 = 2$, $r_2 = 3$, $\alpha = 1$. Blue lines refer to no network coding comparative performance.

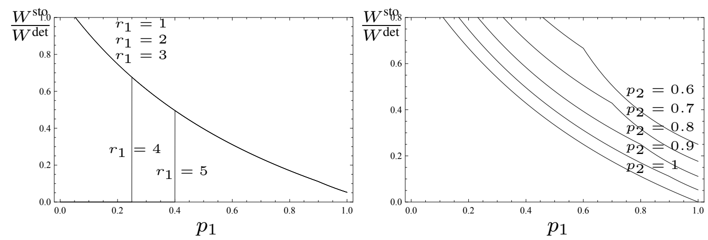

Fig. 6. **Worst-case feedback overhead** varying p_1. Parametric plots vs r_1 (left) and vs p_2 (right). Default parameters: $\lambda_1 = \lambda_2$, $p_2 = 0.9$, $r_1 = 2$, $r_2 = 3$.

and similarly for $\overline{q}_2^{\text{XOR}}$ by exchanging 1 and 2. There are four reasons identified why the stochastic system is more efficient in terms of the number of feedback messages (i.e. overhead).

(i) In the stochastic system NACKs can be used, while in the deterministic ACKs are necessitated. This makes significant difference if the overhearing probabilities p_1, p_2 are close to 1, as is the case where we expect higher throughput benefits.

(ii) If $\lambda_1 p_1 \neq \lambda_2 p_2$, some of the bad packets are transmitted natively, thus $q_i^{\text{XOR}} < 1$ for some i.

(iii) When the source nodes have multiple neighbors, reports from neighbors that are not useful to the coding node are avoided in the stochastic system.

(iv) If (λ_1, λ_2) is in the interior of the region without coding, then the overhead for the stochastic is very small.

In what follows, we study the example of Fig. 1 where $\mathcal{N}_1 = \{R, 2\}$ and $\mathcal{N}_2 = \{R, 1\}$, in which case the best performance of the deterministic system is obtained with respect to (iii) above. Also, we calculate the worst case performance for the stochastic as regards (iv).

In Fig. 5, 6, we present performance plots for *throughput efficiency* $\frac{\lambda^\star_{\text{sto}}}{\lambda^\star_{\text{det}}}$ defined as the ratio of the maximum sum throughput $\lambda_1^\star(\alpha) + \lambda_2^\star(\alpha)$ for the two systems and *worst-case feedback overhead* $\frac{W^{\text{sto}}}{W^{\text{det}}}$ defined as the ratio of average messaging rate calculated on the boundary of the stochastic system region. In Fig. 5. we observe that the throughput drops significantly when p_f is small, i.e. when the fast flow has weak overhearing channel (see the case for $r_1 = 5$ in the left). In all other cases, the stochastic system sacrifices only a small fraction of the throughput (stays always above 95%). In Fig. 6, the corresponding gain in overhead is shown. In the left plot, where $p_2 = 0.9$, we see that the stochastic system requires at most 50% of the messages used in the deterministic one (if $r_1 = 5$), while, independently of rate, this figure becomes as small as 5% if the probabilities are both high (e.g. for $p_1 = p_2 = 0.9$), which is the most practical case. In the right plot we verify that the stochastic system is not a good option when both probabilities have middle values.

VII. Conclusion

In the problem of reporting overhearing events in wireless network coding, we study the deterministic system and the stochastic one. We derive analytical expressions for the throughput region of the first and the code-constraint region of the second and we show that the two are equal when $r_1 = r_2$. When $r_1 \neq r_2$, we analyze the throughput-overhead tradeoff and conclude that the stochastic system is a very efficient approach when the overhearing probabilities are sufficiently high. Alongside with the theoretical results, we propose simple and efficient evacuation policies which can be used in practice to achieve optimal throughput for the case of two receivers or good performance for more than two receivers.

Appendix

Proof of Theorem 7: We assume that the packets are served from the queues in a FCFS manner, since all packets in a given queue are statistically equivalent and thus reordering them does not change the expected outcome.

We partition the set of policies Π to three sets, the subset of policies using only single controls Π_{sin}, the subset of policies using always XOR controls if $\mathcal{C}^x \cap \mathcal{C}_{sto}(t) \neq \emptyset$, called Π_{xor} and the rest Π_{mix}. We immediately get

$$\overline{T}^{\pi}(k_1, k_2) \geq \frac{k_1}{r_1} + \frac{k_2}{r_2}, \text{ for all } \pi \in \Pi_{sin}. \tag{14}$$

Next we will find a bound for policies in Π_{xor} and ultimately we will show that the policies in Π_{mix} are outperformed (in asymptotic sense) by those in $\Pi_{sin} \cup \Pi_{xor}$.

Let $N_{\min} \triangleq \min\{N_1, N_2\}$ and recall $(f, s) = (1, 2)$ if $r_1 \geq r_2$ and $(f, s) = (2, 1)$ otherwise. Observe that the following hold under any policy in Π_{xor}:

1) While XOR controls are still available (i.e. $\mathcal{C}^x \cap \mathcal{C}_{sto}(t) \neq \emptyset$), a good packet departs only if coded with another good packet independently of the XOR control used.
2) At the end of the slot that the packets from one flow are all evacuated for the first time, it holds: exactly $N_{\min}$ good packets of both flows have departed.

We make the following helpful conventions:

1) In case of a $\{u_1 \oplus u_2\}$ control involving two bad packets followed by a single control of one of the two bad packets (the combination evacuates both packets), we assign one evacuated packet to each control.
2) Then, all XOR controls evacuate exactly one packet with the exception of the control $\{g_1 \oplus g_2\}$, which evacuates two packets. We make the convention that the first $N_{\min}$ good packets of the fast flow take up zero transmissions (the corresponding transmissions are counted for the first $N_{\min}$ good packets of the slow flow).

Let $J(i) - 1, i = 0, 1$ be the number of packets in front of the $N_{\min} + i$-th good packet in the unknown queue of the fast flow at time 0. Using the law of iterative expectations we get $\mathbb{E}[J(0)] = \mathbb{E}[N_{\min}]/p_f$ and $\mathbb{E}[J(1)] = (\mathbb{E}[N_{\min}] + 1)/p_f$.

All packets of the slow flow plus the bad packets of fast flow of at least up to $J(0)$ are evacuated in slots of r_s packets requiring one transmission each. Then the remaining $k_f - J(0)$ packets of the fast flow are evacuated in slots of r_f packets. Thus, for any $\pi \in \Pi_{XOR}$

$$\begin{aligned}
\overline{T}^{\pi} &\geq \mathbb{E}\left[\left\lceil \frac{k_s + J(0) - N_{\min}}{r_s} \right\rceil\right] + \mathbb{E}\left[\left\lceil \frac{k_f - J(0)}{r_f} \right\rceil\right] \\
&\geq \mathbb{E}\left[\frac{k_s + J(0) - N_{\min}}{r_s}\right] + \mathbb{E}\left[\frac{k_f - J(0)}{r_f}\right] \\
&= \frac{k_s}{r_s} + \frac{(1-p_f)\mathbb{E}[N_{\min}]}{p_f r_s} + \frac{k_f}{r_f} - \frac{\mathbb{E}[N_{\min}]}{p_f r_f} \\
&= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{\max\{r_1, r_2\}} - \frac{1-p_f}{\min\{r_1, r_2\}}\right],
\end{aligned} \tag{15}$$

which combined with (14) yields

$$\overline{T}^{\pi}(k_1, k_2) \geq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{r_f} - \frac{1-p_f}{r_s}\right]^+,$$

for all $\pi \in \Pi_{sin} \cup \Pi_{xor}$. Using $\lim_{t\to\infty} \frac{\mathbb{E}[N_{\min}]}{t} = \min\{k_1 p_1, k_2 p_2\}$ found above, we conclude that

$$\liminf_{t\to\infty} \frac{\overline{T}^{\pi}(\lceil t\lambda_1 \rceil, \lceil t\lambda_2 \rceil)}{t} \geq B^{req}, \quad \pi \in \Pi_{sin} \cup \Pi_{xor},$$

where B^{req} is the requested limit. Next, we consider set Π_{mix}.

Pick a policy $\pi \in \Pi_{mix}$. Let $L_s(k_s, k_f, \omega), L_f(k_s, k_f, \omega)$ be random variables denoting the number of packets that were evacuated with controls $\{g_s\}, \{u_s\}$ and $\{g_f\}, \{u_f\}$ respectively. We have $l_i \triangleq \mathbb{E}[L_i(k_s, k_f, \omega)]$ and $0 \leq l_i \leq k_i$, for $i \in \{s, f\}$.

Let $M_s(k_s, k_f, \omega), M_f(k_s, k_f, \omega)$ be the number of good packets that were evacuated with the above controls in the fast and slow flow respectively. Furthermore, let $H_i(k_s, k_f, \omega)$ be the number of good packets evacuated by controls $\{g_i\}$. By the law of large numbers we have w.p.1:

$$\lim_{t\to\infty} \frac{M_i(k_s t, k_f t, \omega)}{t} = \mathbb{E}[H_i] + p_i(\mathbb{E}[L_i] - \mathbb{E}[H_i]) \geq p_i l_i. \tag{16}$$

All k_s packets and the $k_f - L_f$ packets of the fast flow are evacuated with rate r_s. Therefore, the expected number of timeslots needed to evacuate these packets is:

$$\begin{aligned}
\overline{T}_1 &\geq \mathbb{E}\left[\frac{k_s}{r_s} + \frac{k_f - L_f}{r_s} - \frac{\min(N_s - M_s, N_f - M_f)}{r_s}\right] \\
&= \frac{k_s}{r_s} + \frac{k_f - l_f}{r_s} - \frac{\mathbb{E}[\min(N_s - M_s, N_f - M_f)]}{r_s},
\end{aligned} \tag{17}$$

where we have subtracted the time corresponding to XORs between good packets. Also, the inequality is due to the assumption that no dummy packets were used. The rest L_f packets are evacuated with rate r_f thus:

$$\overline{T}_2 \geq \mathbb{E}\left[\left\lceil \frac{L_f}{r_f} \right\rceil\right] \geq \frac{l_f}{r_f}. \tag{18}$$

Therefore, using (17) and (18), we have:

$$\overline{T}^{\pi}(k_1,k_2)=\overline{T}_1+\overline{T}_2\geq\frac{k_s}{r_s}+\frac{k_f}{r_s}-l_f\left(\frac{1}{r_s}-\frac{1}{r_f}\right)\\-\frac{\mathbb{E}[\min(N_s-M_s,N_f-M_f)]}{r_s}\quad(19)$$

Define $B^{mix}\triangleq\liminf_{t\to\infty}\frac{\overline{T}^{\pi}(\lceil tk_1\rceil,\lceil tk_2\rceil)}{t}$, $\pi\in\Pi_{mix}$. Taking the limit in RHS of (19), using uniform integrability of the considered random sequences, we get w.p.1:

$$B^{mix}\overset{(16)}{\geq}\frac{k_s}{r_s}+\frac{k_f}{r_s}-l_f\left(\frac{1}{r_s}-\frac{1}{r_f}\right)\\-\frac{\min\left(p_s(k_s-l_s),p_f(k_f-l_f)\right)}{r_s}.\quad(20)$$

Next we show that $B^{mix}\geq B^{req}$. Define the conditions:

$$c_1\equiv p_sk_s\geq p_fk_f\qquad c_2\equiv p_s(k_s-l_s)\geq p_f(k_f-l_f)$$

Using $0\leq l_i\leq k_i$, $i\in\{s,f\}$, for $\overline{p_f}>\frac{r_s}{r_f}$ we have

$$B^{mix}-B^{req}=\\ \begin{cases}(k_f-l_f)(\frac{\overline{p_f}}{r_s}-\frac{1}{r_f}) & ,c_2\\ \frac{p_f(k_f-l_f)-p_s(k_s-l_s)}{r_s}+(k_f-l_f)(\frac{\overline{p_f}}{r_s}-\frac{1}{r_f}) & ,\overline{c_2},\end{cases}$$

while for $\overline{p_f}\leq\frac{r_s}{r_f}$

$$B^{mix}-B^{req}=\\ \begin{cases}l_f(\frac{1}{r_f}-\frac{\overline{p_f}}{r_s}) & ,c_1\text{ and }c_2\\ \frac{p_f(k_f-l_f)-p_s(k_s-l_s)}{r_s}+l_f(\frac{1}{r_f}-\frac{\overline{p_f}}{r_s}) & ,c_1\text{ and }\overline{c_2}\\ \frac{p_sk_s+p_fl_f-p_fk_f}{p_f}(\frac{1}{r_f}-\frac{\overline{p_f}}{r_s}) & ,\overline{c_1}\text{ and }c_2\\ \frac{p_f(k_f-l_f)-p_s(k_s-l_s)}{p_f}(\frac{1}{r_s}-\frac{1}{r_f}) & \\ +\frac{p_sl_s}{p_f}(\frac{1}{r_f}-\frac{\overline{p_f}}{r_s}) & ,\overline{c_1}\text{ and }\overline{c_2}\end{cases}$$

All cases can be verified to be nonnegative. ■

Proof of Theorem 8: We follow the steps of the proof of Theorem 7 closely. First, note that if $1-p_f>\frac{\min(r_1,r_2)}{\max(r_1,r_2)}$ is true, then π^* chooses only single controls and we quickly get

$$\hat{T}^{\pi^*}(\lambda_1,\lambda_2)=\frac{\lambda_1}{r_1}+\frac{\lambda_2}{r_2}.$$

If on the other hand the condition is false, then we have $\pi^*\in\Pi_{XOR}$. The difference from the proof of Theorem 7 is how packets between $J(0)$ and $J(1)$ are treated.

$$\overline{T}^{\pi^*}(k_1,k_2)\leq\mathbb{E}\left[\left\lceil\frac{k_s+J(1)-N_{\min}}{r_s}\right\rceil\right]+\mathbb{E}\left[\left\lceil\frac{k_f-J(0)}{r_f}\right\rceil\right]\\ \leq\mathbb{E}\left[\frac{k_s+J(1)-N_{\min}}{r_s}\right]+\mathbb{E}\left[\frac{k_f-J(0)}{r_f}\right]+2\\ =\frac{k_s}{r_s}+\frac{(1-p_f)\mathbb{E}[N_{\min}]}{p_fr_s}+\frac{1}{p_fr_s}+\frac{k_f}{r_f}-\frac{\mathbb{E}[N_{\min}]}{p_fr_f}+2\\ =\frac{k_1}{r_1}+\frac{k_2}{r_2}-\frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{\max\{r_1,r_2\}}-\frac{1-p_f}{\min\{r_1,r_2\}}\right]\\ +2+\frac{1}{p_f\min\{r_1,r_2\}}.$$

Taking the $\limsup$ completes the proof. ■

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