Vacation Policy Optimization with Application to IEEE 802.16e Power Saving Mechanism

A. P. Azad*, S. Alouf*, E. Altman*, V. Borkar† and G. Paschos+
* Maestro group, INRIA, 2004 Route des Lucioles, F-06902 Sophia Antipolis
†School of Technology, TIFR, Mumbai
+ ECE @ University of Thessaly, Volos, Greece
{aazad,salouf,altman}@sophia.inria.fr, borkar@tifr.res.in, gpasxos@uth.gr

***Abstract*—Much research has been devoted to optimizing the power saving mechanism in wireless mobile devices. Recent advances in wireless radio technology facilitate the implementation of various possible sleep policies. One basic question arises, which policy performs best in a given condition. Furthermore, what are the optimal parameters for a given policy. To answer these questions, we provide a formulation of an optimization problem, which entails cost minimization for a given parameterized policy and selection of the best policy among a class. We propose a cost function which captures the inherent tradeoff of delay and energy saving. This takes into account the cost of response time due to extra sleep, the energy saving during the sleep, and the cost for periodic waking up for listening. As an application, we consider IEEE 802.16e sleep power saving mechanism. We derive various practical policies and check their performances. We show that constant duration policy is optimal for exponentially distributed inactive durations. We also show that this structure is not optimal for hyper-exponentially distributed inactive periods. Several extra structural and numerical results are obtained for analytically intractable policies. Our framework allows us to compare the performance of several optimal and suboptimal practical policies with the proposed standard for WiMAX policy.**

I. Introduction

A wireless device using contemporary radio technology can obtain great energy benefits by shutting off the transceiver whenever there are no communications taking place, a state that is called sleep mode. Assuming that the device is connected to Internet through a gateway (e.g. a base station), the attention of the mobile may be required by an incoming activity. Since the mobile transceiver is shut off, a response delay will be incurred. One then has to be very careful as how to schedule sleep periods in order to minimize energy consumption and reduce delays.

Since the initial announcement of IEEE 802.16e standards for mobility [1], there is an important volume of performance studies on the subject. The first chronologically approach is found in [12]. In an effort to relax some assumptions, [15], [13] study the impact of outgoing traffic, [8], [7] study the effect of setup time while [9], [3] deal with queueing implications in the analysis.

The above models assume a Poisson process for the packet arrivals. In [14], the author is using hyper-Erlang distribution for the packet interarrival period. In [6], [2], hyperexponential arrivals are proposed. In any of the above cases an exogenous arrival process that does not depend on the energy management scheme is considered. Moreover, the delay metric taken is the average packet delay in the system.

Rather than assuming an exogenous independent arrival process, we have in mind elastic arrival processes in which (i) an idle period or *off period* begins when the activity of the mobile ends, and (ii) the duration of the *on period* does not depend on the response delay, defined as the time that elapses between the instant a request is issued and the instant at which the service actually begins. Both assumptions are appropriate to interactive applications such as web browsing. As a result, the measure for delay is taken to be the response delay of the mobile to the oldest activity in the given idle period.

The distribution of the duration of the off period is often modeled as an exponential random variable. We shall also be interested in the case that the parameter of the exponential distribution is unknown but we have a known prior distribution on that parameter. This is equivalent to using a hyper-exponential distribution for the off duration.

In the literature, there are other works that provide evidence of heavy-tailed off period distributions on the Internet and on the World Wide Web with a Pareto type distribution. In [11] the corresponding measured shape parameter is 1.2. In [5] the operator idle periods are found to be heavy-tailed. As heavy-tailed distributed random variables can be well approximated by hyper-exponential distributions [10], [6], we face yet another motivation to study off times with hyper-exponential distributions.

The contributions of this paper are as follows:

- We provide an optimization framework for a large class of practical policies;
- We consider different strategies of power saving, and derive global optimal behavior in different scenarios;
- We show that when the incoming traffic is a Poisson arrival process the optimal strategy is the repeated constant policy;
- We provide numerical results for cost minimization within a class of parameterized sleep policies.
- Finally, the optimal performance is compared to the performance of the standards.

The rest of the paper is structured as follows: Sect. II outlines our system model while section III introduces the cost

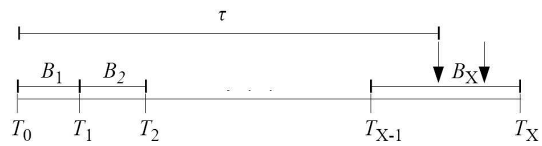


Fig. 1. An idle period (or off period). At time T_i, the mobile decides on a random vacation B_{i+1} and goes to sleep. At time $T_{i+1} = T_i + B_{i+1}$ wakes up to check for activity. The idle period ends when an activity is detected.

function and states the optimization problem that is considered in the rest of the paper. Section IV investigates strategies with identically distributed vacations and section V those with non identically distributed ones. Numerical results and a comparative study of the different optimal strategies and of the IEEE 802.16e standard are reported in Sect. VI, before concluding the paper in Sect. VII.

II. System Model

Consider a mobile wireless device connected to the Internet through a gateway. When the device is idle it requires P_L units of power for keeping the transceiver working, a mode called *listen*. Instead, if the transceiver is turned off, it consumes only P_S units of power, with $P_S < P_L$, while being in *sleep* mode. The device is then eager to go to sleep mode in order to save power and extend battery life. If, however, the device is in sleep mode, an incoming activity (packet or flow) will be stalled at the gateway. The device should frequently turn to listen mode in order to check for such incoming activity.

In each idle period, the device goes through a sequence of sleep and listen modes until an incoming activity is detected, see figure 1. In particular, in the beginning of each sleep period, the device chooses the sleep mode window while the listening mode window is considered fixed and negligible. At the end of the sleep window, the device switches to listen mode. In case there is no incoming activity waiting at the gateway, a fixed energy cost is incurred for checking the system state. In case at least one activity has arrived, the idle period is finished and a delay cost is incurred depending on the waiting time of the first activity. In the spirit of achieving a Quality of Service (QoS) tradeoff, we are interested in finding the optimal policy that minimizes the total cost.

An equivalent modeling of the system is one that considers a server that goes on repeated vacations. The incoming traffic load is replaced by customers waiting to be served by the server. In the rest of the paper we use the notation of a server with vacations as in [3]. The vacation length is then equivalent to sleep mode window.

Let X denote the number of vacations in an idle period, where X is a discrete random variable taking values in $\mathbb{N}^*$. Let τ denote the time between the start of the first vacation and the arrival of a customer; this time is referred to as the idle period. τ is a random variable whose probability density function is $f_\tau(t), t \geq 0$.

The duration of the kth vacation is a random variable denoted B_k, for $k \in \mathbb{N}^*$. In this paper, we consider vacation processes $\{B_k\}_{k\in\mathbb{N}^*}$ that consist of mutually independent

TABLE I
Glossary of notations

X	Number of vacations
τ	Arrival time of first customer
B_k	Duration of kth vacation
b	Parameter of the distribution of B_k
α	Parameter of the distribution of B_k for Scaled General Vacations
$\boldsymbol{p}$	Distribution of B_k for Scaled and General Discrete Vacations
T_k	Time until kth vacation, $T_k = \sum_{i=1}^{k} B_i$
T_0	Starting time of power save mode, $T_0 = 0$
$\mathcal{Y}^*(s)$	Laplace-Stieltjes transform of a random variable Y
E_L	Energy consumed when listening to the channel
P_S	Power consumed by a mobile in a sleep state
ϵ	Normalized energy weight, $0 < \epsilon \leq 1$
$\bar{\epsilon}$	Normalized delay weight, $\bar{\epsilon} = 1 - \epsilon$
V	Cost function
W_{-1}	Branch of the Lambert W function that is real-valued on the interval $[-\exp(-1), 0]$ and always below -1
$\boldsymbol{\lambda}$	rate vector in the n-phase hyper-exponential distribution, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$
$\boldsymbol{q}$	probability vector in the n-phase hyper-exponential distribution $\boldsymbol{q} = (q_1, \ldots, q_n)$
η	$= \bar{\epsilon} + \epsilon P_S, \quad 0 < \eta \leq 1 + P_S$
ζ_i	$= 1 + \frac{\lambda_i \epsilon E_L}{\eta}, \quad i = 1, \ldots, n, \quad \zeta_i > 1$

random variables. The time at the end of the kth vacation is a random variable denoted T_k, for $k \in \mathbb{N}^*$. We denote T_0 as the time at the beginning of the first vacation; by convention $T_0 = 0$. We naturally have $T_k = T_{k-1} + B_k = \sum_{i=1}^{k} B_i$.

We will use the following notation $\mathcal{Y}^*(s) := \mathbb{E}[\exp(-sY)]$ to denote the Laplace-Stieltjes transform of a generic random variable Y evaluated at s. Hence, we can readily write $\mathcal{T}_k^*(s) = \prod_{i=1}^{k} \mathcal{B}_i^*(s)$. Observe that the time at the end of a generic idle period is simply T_X. Hence, since the arrival time of the first customer during the idle period is τ, the service of this customer will be delayed for $T_X - \tau$ units of time.

The energy consumed by a mobile while listening to the channel and checking for customers is denoted E_L. This is actually a penalty paid at the end of each vacation. The energy consumed by a mobile during vacation B_k is then equal to $E_L + P_S B_k$ and that consumed during a generic idle period is equal to $E_L X + P_S T_X$.

For convenience, we have grouped the major notation used in the paper in Table I.

III. Problem Formulation

We are interested in minimizing the cost of the sleep mode in an arbitrary idle period. The cost is the weighted sum of the energy consumed during the period and the *extra* delay incurred on the traffic by the unavailability of the mobile. Let V be this cost, which is written as follows

$$V := \bar{\epsilon}\,\mathbb{E}[T_X - \tau] + \epsilon\left(E_L \mathbb{E}[X] + P_S \mathbb{E}[T_X]\right) \qquad (1)$$

where ϵ is a normalized weight that takes value between 0 and 1, and $\bar{\epsilon} = 1 - \epsilon$.

The distribution of vacation length B_k may depend on state k, in which case we deal with non identically distributed vacations, or it may not depend, in which case we deal with identically distributed vacations. Deterministic vacations are obtained as particular cases. For a given vacation pattern (having a parameterized distribution), our objective is to find the optimal parameters of the distribution that can solve the following minimization problem for a given input process:

$$\min_b V. \tag{2}$$

The variable b above is the parameterized variable of the distribution of B_k. In some cases, we may have a vector instead of b. The following can be directly derived

$$\begin{aligned} \mathbb{E}[T_X - \tau] &= \sum_{k=1}^{\infty} \mathbb{E}[(T_k - \tau)\mathbb{1}\{T_{k-1} < \tau \le T_k\}]; \\ \mathbb{E}[X] &= \sum_{k=1}^{\infty} kP(X = k), \\ \mathbb{E}[T_X] &= \sum_{k=1}^{\infty} \mathbb{E}[T_k]P(X = k); \end{aligned}$$

where, for $k \in \mathbb{N}^*$, $P(X = k) = P(T_{k-1} < \tau \le T_k)$.

The cost can then be rewritten

$$\begin{aligned} V &= \sum_{k=1}^{\infty} \Big\{ \bar{\epsilon}\, \mathbb{E}[(T_k - \tau)\mathbb{1}\{T_{k-1} < \tau \le T_k\}] \\ &\qquad + \epsilon P(X = k) \left(E_L k + P_S \mathbb{E}[T_k]\right) \Big\}. \end{aligned} \tag{3}$$

Hence, to optimize a given sleep mode pattern, we can equivalently minimize (3).

The idle period random variable plays an important role on the above problem. We assume that τ is hyper-exponentially distributed with n phases and parameters $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$. In other words, we have

$$f_\tau(t) = \sum_{i=1}^{n} q_i \lambda_i \exp(-\lambda_i t), \quad \sum_{i=1}^{n} q_i = 1. \tag{4}$$

Recall that τ represents the time that elapses since the beginning of the save mode (i.e., beginning of the first vacation) until the arrival of the first packet (i.e., customer). Therefore, τ is the conditional residual inter-arrival time. Observe that when $n = 1$, τ will be exponentially distributed with parameter λ_1. This case is equivalent to having a Poisson arrival process with rate λ_1, thanks to the memoryless property.

We will now compute the elements of (3) when τ is hyper-exponentially distributed. The distribution of X is given as

$$\begin{aligned} P(X = k) &= P(\tau > T_{k-1}) - P(\tau > T_k) \\ &= \sum_{i=1}^{n} q_i \mathbb{E}[e^{-\lambda_i T_{k-1}}] - \sum_{i=1}^{n} q_i \mathbb{E}[e^{-\lambda_i T_k}] \\ &= \sum_{i=1}^{n} q_i \mathcal{T}_{k-1}^*(\lambda_i) \left(1 - \mathcal{B}_k^*(\lambda_i)\right). \end{aligned}$$

Also, we can compute

$$\begin{aligned} &\mathbb{E}[(T_k - \tau)\mathbb{1}\{T_{k-1} < \tau \le T_k\}] \\ &\quad = \mathbb{E}[(T_{k-1} + B_k - \tau)\mathbb{1}\{T_{k-1} < \tau \le T_{k-1} + B_k\}] \\ &\quad = \sum_{i=1}^{n} q_i \mathcal{T}_{k-1}^*(\lambda_i) \left(\mathbb{E}[B_k] - \frac{1 - \mathcal{B}_k^*(\lambda_i)}{\lambda_i} \right). \end{aligned} \tag{5}$$

After some calculus, the cost simplifies to (denote $\eta = \bar{\epsilon} + \epsilon P_S$)

$$V = -\bar{\epsilon}\, \mathbb{E}[\tau] + \sum_{k=0}^{\infty} \sum_{i=1}^{n} q_i \mathcal{T}_k^*(\lambda_i) \left(\epsilon E_L + \eta \mathbb{E}[B_{k+1}]\right), \tag{6}$$

where $\mathbb{E}[\tau] = \sum_{i=1}^{n} q_i/\lambda_i$ is the expectation of τ.

Finally, we formulate an extra optimization problem. Intuitively, we would like to obtain the best performance of sleeping that is possible for a mobile under a given "off period" distribution. We constrain, however, the problem to the class of policies that have discrete and independent vacations. This is a problem that stems from reconciling the practical application of the sleep policy and the tractability of the approach.

$$\min_{\{B_k(b^\star)\}_{k\ge 1}} \left\{ \min_b V \right\}. \tag{7}$$

The final problem is then defined for identically distributed vacations as "given a class of distributions for $\{B_k\}$, find the distribution that provides the minimum cost". We will study several patterns of the vacations $\{B_k\}_{k\in\mathbb{N}^*}$ and try to minimize the cost V for each case. In section VI, we will provide a numerical solution to the problem (7).

If we relax the constraint of identically distributed vacations, the mobile is free to choose any vacation distribution at each waking up instant, a fact that complexes the problem immensely. We will narrow the problem by considering only deterministic vacations in this case.

IV. Identically Distributed Vacations

We assume in this section that all vacations are identically distributed, in other words, the control is static. Let B be a generic random variable having same distribution as any of the vacations. Equation (6) can be rewritten

$$V = -\bar{\epsilon}\, \mathbb{E}[\tau] + (\epsilon E_L + \eta \mathbb{E}[B]) \sum_{i=1}^{n} \frac{q_i}{1 - \mathcal{B}^*(\lambda_i)}. \tag{8}$$

In the rest of this section, we will propose different strategies and derive the optimal control in each case. The strategies that are considered are

- Exponentially distributed vacations; the parameter to optimize is the expectation of B;
- Equally sized vacations (periodic pattern); the parameter to optimize is the constant vacation;
- General vacations that follow a scaled version of a known distribution; the parameter to optimize is the scale;
- General discrete vacations; the parameter to optimize is the distribution itself.

A. Exponentially Distributed Vacations

The first strategy considered is when vacations are i.i.d. exponential random variables with mean $\mathbb{E}[B] = b$, i.e., $\mathcal{B}^*(s) = 1/(1 + bs)$ for $k \in \mathbb{N}^*$. Performance is optimized by controlling the mean vacation size b. The cost is denoted $V_e(b)$, where the subscript stands for "exponential"; it simplifies to (recall that $\eta = \bar{\epsilon} + \epsilon P_S$)

$$\begin{aligned} V_e(b) &= -\bar{\epsilon}\,\mathbb{E}[\tau] + (\epsilon E_L + \eta b) \sum_{i=1}^{n} q_i \left(1 + \frac{1}{b\lambda_i}\right), \\ &= \epsilon \left(P_S + \frac{E_L}{b}\right) \mathbb{E}[\tau] + (\epsilon E_L + \eta b). \end{aligned} \tag{9}$$

Proposition 4.1: The cost $V_e(b)$ is a convex function having a minimum at

$$b_e^\star = \sqrt{\frac{\epsilon E_L \mathbb{E}[\tau]}{\eta}} = \sqrt{\frac{\epsilon E_L \mathbb{E}[\tau]}{\bar{\epsilon} + \epsilon P_S}}. \tag{10}$$

The minimal cost is

$$V_e(b_e^\star) = \epsilon(P_S \mathbb{E}[\tau] + E_L) + 2\sqrt{\epsilon \eta E_L \mathbb{E}[\tau]} \tag{11}$$

Proof: We refer appendix A for detailed proof. ◊

The optimal expected vacation duration and the minimal cost are obtained in closed-form.

Remark 4.1: The case of Poisson arrivals with rate λ_1, which is equivalent to having $n = 1$, leads to the same optimal control $b_e^\star$ with $\mathbb{E}[\tau] = 1/\lambda_1$.

Remark 4.2: The optimal mean vacation in eq. (22) and optimal minimal cost in eq. 23 stands valid for any distribution of Off times τ for a known mean of Off time $\mathbb{E}[\tau]$.
This is direct from eq. (3). Observe that in eq. (3) the term $\mathbb{E}[(T_k - \tau)\mathbb{1}\{T_{k-1} < \tau \leq T_k\}]$ is exponentially distributed due to memoryless property of vacations. Hence it is valid for any distribution of Off times τ for a known $\mathbb{E}[\tau]$.

B. Equally Sized Vacations

The second strategy considered is that where vacations are constant, in other words, $B = b$. The performance is optimized by controlling the size of b. The Laplace-Stieltjes transform of B becomes $\mathcal{B}^*(s) = \exp(-sb)$, yielding the following simplified expression for the cost (the subscript stands for "constant")

$$V_c(b) = -\bar{\epsilon}\mathbb{E}[\tau] + (\epsilon E_L + \eta b) \sum_{i=1}^{n} \frac{q_i}{1 - \exp(-\lambda_i b)}. \tag{12}$$

Proposition 4.2: When $n = 1$, the cost $V_c(b)$ is a convex function having a minimum at

$$b_c^\star = -\frac{1}{\lambda_1}\left(\zeta_1 + W_{-1}(-e^{-\zeta_1})\right) \tag{13}$$
$$\text{with } \zeta_1 := \frac{\lambda_1 \epsilon E_L}{\eta} + 1,$$

where $W_{-1}(-e^{-\zeta_1})$ denotes the branch of the Lambert W function[1] that is real-valued on the interval $[-\exp(-1), 0]$ and always below -1. The minimal cost is

$$V_c(b_c^\star) = -\frac{1}{\lambda}\left(\bar{\epsilon} + \eta W_{-1}(-e^{-\zeta_1})\right). \tag{14}$$

[1]The Lambert W function, satisfies $W(x)\exp(W(x)) = x$. As the equation $y\exp(y) = x$ has an infinite number of solutions y for each (non-zero) value of x, the function $W(x)$ has an infinite number of branches.

Proof: We refer appendix B for detailed proof. ◊

Proposition 4.3: The cost $V_c(b)$ is a convex function having a minimum in $]0, \infty[$.

Proof: We refer appendix C for detailed proof. ◊

Proposition 4.3 proves the existence of a global minimum. Unfortunately, we are not able to derive the optimal $b_c^\star$ analytically and use numerical methods to find $b_c^\star$. The dimensionality of the problem can be showcased by the following result.

Proposition 4.4: When $n > 1$, no optimal policy can be independent of $\boldsymbol{q} = (q_1, \ldots, q_n)$.

Proof: We refer appendix C for detailed proof. ◊

C. Scaled General Vacations

In this third strategy, we consider the random vacation B to be a factor α of a random variable S with a general distribution, i.e. $B = \alpha S$. For a given distribution of S, the scaling factor α is controlled to optimize the performance. We can readily write $\mathcal{B}^*(s) = \mathcal{S}^*(\alpha s)$ and $\mathbb{E}[B] = \alpha\mathbb{E}[S]$. We denote the cost as $V_s(\alpha)$ where the subscript stands for "scaled". Hence,

$$V_s(\alpha) = -\bar{\epsilon}\mathbb{E}[\tau] + (\epsilon E_L + \eta\alpha\mathbb{E}[S]) \sum_{i=1}^{n} \frac{q_i}{1 - \mathcal{S}^*(\alpha\lambda_i)}.$$

We consider now that S is a discrete random variable taking values in a finite set $\{a_j\}_{j=1,\ldots,J}$ with a probability distribution $\boldsymbol{p} = (p_1, \ldots, p_J)$, i.e., $P(S = a_j) = p_j$ and $\sum_{j=1}^{J} p_j = 1$. Hence, $\mathcal{S}^*(s) = \sum_{j=1}^{J} p_j \exp(-s a_j)$, and $\mathbb{E}[S] = \sum_{j=1}^{J} p_j a_j$.

The strategy advocates to have each vacation follow a discrete general distribution, taking values in $\{\alpha a_j\}_{j=1,\ldots,J}$. The probability distribution $\boldsymbol{p}$ is assumed fixed whereas the set of possible values can be scaled for minimal cost.

The optimization problem can be stated as

$$\min_{\alpha} V_s(\alpha) \quad \text{subject to } \alpha > 0. \tag{15}$$

The optimal policy is $\alpha^\star = \arg\min_\alpha V_s(\alpha)$. It is intractable to solve analytically 15, we will therefore resort to a numerical resolution.

D. General Discrete Vacations

The fourth strategy resembles the third one in that it equally considers a discrete general vacation for the variable B. However, the set of possible values is now fixed (i.e., $c = 1$) whereas the probability distribution $\boldsymbol{p}$ can be optimized for minimal cost. We denote the cost as $V_g(\boldsymbol{p})$, where the subscript stands for "general", and write

$$V_g(\boldsymbol{p}) = -\bar{\epsilon}\mathbb{E}[\tau] + \sum_{i=1}^{n} \frac{q_i\left(\epsilon E_L + \eta \sum_{j=1}^{J} p_j a_j\right)}{1 - \sum_{j=1}^{J} p_j \exp(-\lambda_i a_j)}.$$

Our objective is to find $\boldsymbol{p}^\star = \arg\min_{\boldsymbol{p}} V_g(\boldsymbol{p})$ such that $0 \leq p_j \leq 1$ for $j = 1, \ldots, J$ and $\sum_{j=1}^{J} p_j = 1$. This optimization problem will be solved numerically.

V. Non Identically Distributed Vacations

In this section we consider vacations that are not identically distributed. We restrain ourselves to the set of policies in which vacations are deterministic.

Let the kth vacation be of fixed size b_k. The instants $\{T_k\}_{k\in\mathbb{N}}$ are now deterministic, and we let $t_k = T_k$ for any k to reflect this. We have $t_0 = 0$ and $t_k = \sum_{j=1}^{k} b_j$.

We will next present a particular strategy in which vacations increase in size over time, then move to the general deterministic case and show some properties of the optimal policy.

A. Vacations Increasing over Time

This strategy is inspired by the power save mode of the IEEE 802.16e [1], more precisely, by type I power saving classes. There, the size of a sleep window (i.e., a vacation) is doubled over time until a maximum permissible sleep window, denoted $b_{\max}$. The size of the kth vacation is then

$$b_k = b_1 2^{\min\{k-1,l\}}, \quad k \in \mathbb{N}^*$$

where $l := \log_2(b_{\max}/b_1)$. We also have

$$t_k = b_1 \left(\frac{1}{2^{\min\{k,l\}} - 1} + 2^l (k-l)\mathbb{1}\{k > l\} \right), \quad k \in \mathbb{N}^*.$$

Observe that the cost of the power save mode of the IEEE 802.16e standard can be derived from (6), and can be written as

$$V_{\text{Std}} = -\bar{\epsilon}\,\mathbb{E}[\tau] + \sum_{k=0}^{\infty}\sum_{i=1}^{n} q_i e^{-\lambda_i t_k} \left(\epsilon E_L + \eta b_1 2^{\min\{k,l\}} \right). \tag{16}$$

Instead of doubling the vacations over time, we consider a strategy that increases the vacations by a multiplicative factor f (in the standard, $f = 2$). The performance is then optimized by controlling the factor f. In this strategy, we have

$$b_k = b_1 f^{\min\{k-1,l\}}, \quad k \in \mathbb{N}^*$$

$$t_k = b_1 \left(\frac{f-1}{f^{\min\{k,l\}} - 1} + f^l (k-l)\mathbb{1}\{k > l\} \right), \quad k \in \mathbb{N}^*$$

$$V_m(f) = -\bar{\epsilon}\mathbb{E}[\tau] + \sum_{k=0}^{\infty}\sum_{i=1}^{n} q_i e^{-\lambda_i t_k} \left[\epsilon E_L + \eta b_1 f^{\min\{k,l\}} \right] \tag{17}$$

$$f^\star = \arg\min_{f>1} V_m(f). \tag{18}$$

The optimal $f^\star$ and the minimal cost $V_m(f^\star)$ (the subscript stands for "multiplicative") will be computed numerically.

B. General Deterministic Vacations

In this section, no particular pattern is imposed on the vacations. We denote the cost as $V_d(\boldsymbol{\mu})$ where the subscript stands "deterministic" and $\boldsymbol{\mu} := (b_1, b_2, \ldots)$ is the deterministic policy. The cost (6) is rewritten as follows

$$V_d(\boldsymbol{\mu}) = -\bar{\epsilon}\mathbb{E}[\tau] + \sum_{k=0}^{\infty}\sum_{i=1}^{n} q_i \exp(-\lambda_i t_k)\left(\epsilon E_L + \eta b_{k+1}\right). \tag{19}$$

A necessary condition for the existence of an optimal control sequence $\boldsymbol{\mu}^\star = (b_1^\star, b_2^\star, \ldots)$ is that $\mathbf{grad}\, V_d(\boldsymbol{\mu}^\star) = 0$. Our next step is then to compute the partial derivatives. We have, for $j \in \mathbb{N}^*$,

$$\frac{\partial V_d(\boldsymbol{\mu})}{\partial b_j} = \sum_{i=1}^{n} \eta q_i \left[e^{-\lambda_i t_{j-1}} - \sum_{k=j}^{\infty} \lambda_i e^{-\lambda_i t_k} \left(b_{k+1} + \frac{\epsilon E_L}{\eta} \right) \right]. \tag{20}$$

Proposition 5.1: When $n > 1$, no optimal policy can be independent of $\boldsymbol{q} = (q_1, \ldots, q_n)$.

Proof: We refer appendix C for detailed proof. ♢

Remark 5.1: We actually do not know how many minima the multivariate function $V_d(\boldsymbol{\mu})$ has. Should there be only one minimum, then for sure (25) is the unique optimal control, and the optimal deterministic pattern with Poisson arrivals is surely periodic (constant policy).

Even if there were more minima, our intuition says that optimally vacations should all be equally sized due to the memoryless property. Indeed, at the end of any vacation, the mobile "forgets" about the past history and selects the new vacation size solely based on the arrival rate. Therefore, the decision will always be the same.

To have a formal proof of this intuition, one can exploit a dynamic programming approach which is out of the scope of this paper.

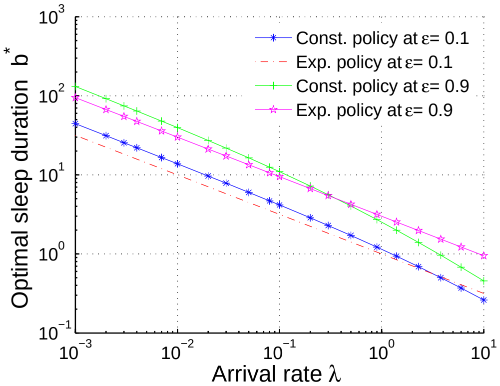
VI. Numerical Investigation

In this section we show some numerical results of our model. We study power saving scheme performance of the IEEE 802.16e standard, which allows mobile terminals to go on sleep mode when there are no packets to serve. Our model is powerful enough to analyze power save mode for large range of wireless systems.

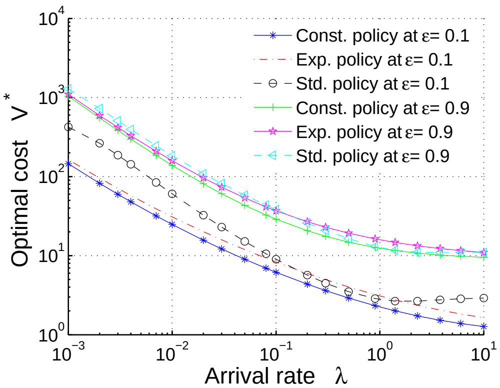
In practice, the mobile needs to check for any packet arrived while it was off. So at the end of each sleep duration, it will awaken and listen to the channel for any notification message from the base station. The mobile needs to switch on the radio every time it wakes up and then listen to the channel for a (small) fixed-duration listening period. Hence, each waking up costs E_L amount of energy consumption. The rate of energy consumption during sleep, P_S, is considerably less than that during listening.

The cost V, defined in (1), captures the main performance measures: energy consumed during the sleep duration and extra delay incurred due to the sleep mode. The cost V is a weighted sum of both metrics. From (1), it comes that a large value of ϵ makes V more sensitive to the energy consumption than to the extra delay, whereas a small ϵ gives more weight to the delay.

The various policies discussed in previous sections are now evaluated numerically. Policies are compared considering as performance metrics the optimal weighted cost and the corresponding optimal sleep duration. In addition, we quantify the performance of the policies devised in this paper by looking at the relative improvement with respect to the IEEE 802.16e protocol. We define the improvement ratio, denoted

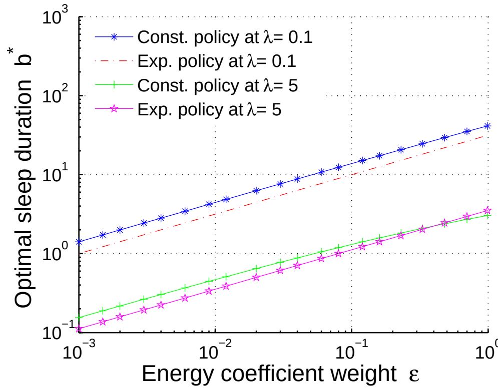


(a) Optimal sleep duration

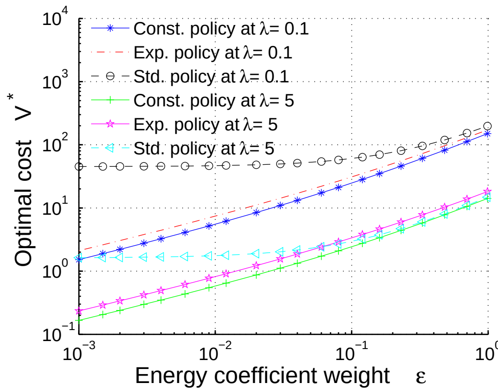


(b) Optimal Cost

Fig. 2. Impact of λ on optimal and standard policies.



(a) Optimal sleep duration



(b) Optimal cost

Fig. 3. Impact of ϵ on optimal and standard policies.

I, as follows:

$$I := \frac{V_{\text{Std}} - V_{\text{Optimal}}}{V_{\text{Std}}}. \tag{21}$$

The physical parameters are set to the following values: $E_L = 10$, and $P_S = 1$. The parameters of the Standard are $b_1 = 2$ and $l = 10$.

A. *Exponential Off Time*

This section reports the comparative study of optimal policies and standard policy (IEEE 802.16e) when arrivals form a Poisson process with rate λ. Three sleep policies are evaluated and compared. Their details are summarized in Table II.

TABLE II
POLICIES USED FOR COMPARISON

Policy	Optimal cost	Control and its optimal value
Exponential	(23)	expected sleep duration, (22)
Constant	(26)	size of fixed sleep duration, (25)
Standard	(16), $n = 1$	–

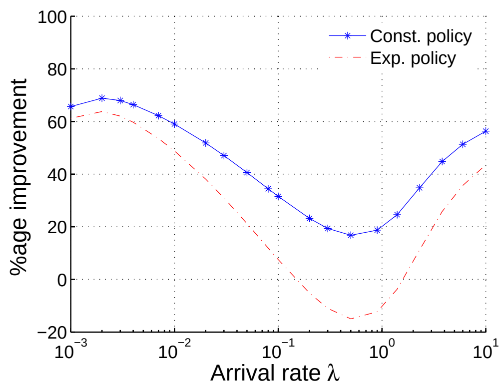
The performance metrics at hand depend on the rate λ and on the normalized weight ϵ. In the following evaluation, we will alternatively vary one of the parameters and fix the other.

The optimal sleep duration for the constant policy and the optimal expected sleep duration for the exponential policy are reported graphically in Fig. 2(a) against the arrival rate λ for two values of ϵ, 0.1 and 0.9. The weight ϵ equal to 0.1 mimics the situation when energy consumption is given higher priority over delay, while ϵ equal to 0.9 mimics the opposite situation. Observe that both optimal parameters decrease as the arrival rate increases. The physical explanation for that is that a larger arrival rate forces the server to be available after shorter breaks, otherwise the cost is too high.

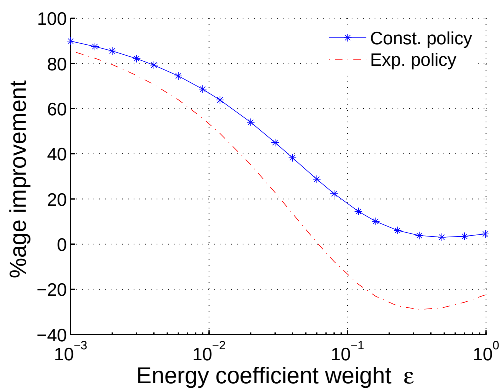
Figure 2(b) depicts the optimal and standard costs obtained under the three policies against the rate λ and for the same values of ϵ. The optimal costs correspond to the optimal parameters plotted in Fig. 2(a). The most relevant observation on Fig. 2(b) is that the optimal constant policy yields better performance than the standard and the exponential policies. This observation supports the findings of Sect. **??**, namely, that the constant policy is the optimal among all possible policies.

Observe also how the cost decreases asymptotically to ϵE_L (1 for $\epsilon = 0.1$ and 9 for $\epsilon = 0.9$) as the rate λ increases. The same trend is observed for the cost of the standard policy. As λ decreases, the increase in both optimal costs is due to the increase of the optimal sleep duration (or of its expectation), while for standard policy it is due to the extra (useless and costly) listening.

We next fix λ to either 0.1 or 5, and vary ϵ. Intuitively, a smaller value of ϵ makes the extra delay more penalizing,

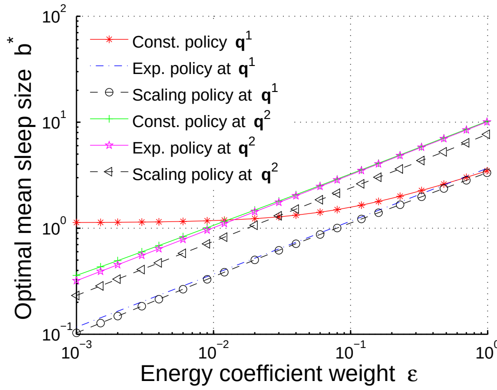


(a) improvement at $\epsilon = 0.1$

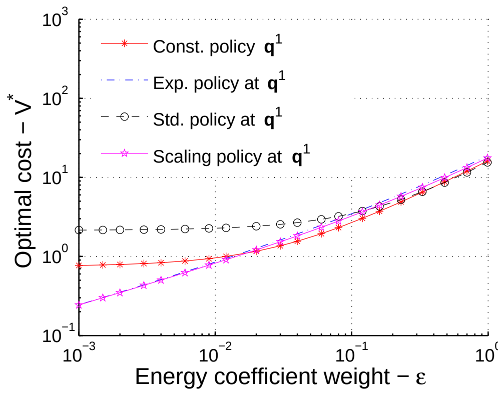


(b) improvement at $\lambda = 0.8$

Fig. 4. Percentage improvement over standard policy.



(a) Optimal expected sleep duration



(b) Optimal cost

Fig. 5. Impact of ϵ on optimal and standard (IEEE 802.16e) policy for hyper-exponential τ at $\boldsymbol{\lambda} = [0.01, 2, 10]$.

enforcing then smaller optimal sleep durations (constant and optimal policy). This is observed in Fig. 3(a). As mentioned earlier, smaller optimal sleep durations yield smaller optimal costs. Thus, the optimal costs increase as ϵ increase, as can be observed in Fig. 3(b). For $\epsilon < 0.1$, the cost of the standard policy is fairly insensitive. This is because the standard has been designed to favor delay over energy: the first sleep duration is small ($b_1 = 2$) and it takes a while before the sleep duration becomes penalizing in terms of delay. This is confirmed by the sudden increase in cost as $\epsilon \geq 1$ (notice the logarithmic scale): when energy consumption costs start to have more weight, the standard policy's performance degrades.

From Figs. 2(b) and 3(b) and Sect. **??**, it is clear that the constant policy is the best. But what about the exponential policy? It seems that it outperforms the standard policy in many cases but not always. To better illustrate this, we compute the percentage improvement I for both constant and exponential policies. Figures 4(a) and 4(b) display I against λ and ϵ, respectively. Interestingly enough, there is a minimum and maximum improvement. Naturally, we always have a positive improvement with the constant policy. Of more interest is the observation that the exponential policy yields substantial improvement over a large range of values of λ and ϵ.

B. Hyper Exponential Off Time

In this section, we consider the situation in which the off time follows an n-phase hyper-exponential distribution.

First, we make a comparative study between the exponential, constant, scaling (cf. Sect. IV-C) and standard policies just like what was done in the previous section. The details provided in Table II hold except that the optimal cost with the constant policy is obtained by minimizating (24). The optimal scaling factor and the optimal cost under the "scaling" policy are derived numerically from (15).

For this study, we consider two distinct distributions with $n = 3$ and $\boldsymbol{\lambda} = [0.01, 2, 10]$ but with different $\boldsymbol{q}$ values so as to have different traffic regimes. The probabilities are $\boldsymbol{q}^1 = [0.1, 0.3, 0.6]$ and $\boldsymbol{q}^2 = [0.6, 0.3, 0.1]$ yielding expected off times equal to 10.21 (moderate incoming traffic) and 60.16 (low incoming traffic), respectively. These values of $\boldsymbol{q}^1, \boldsymbol{q}^2$ have been intentionally chosen so as to show different behaviour of the policies. The parameters of the scaling policy are (distribution of the variable S) $\{a_1, a_2, a_3\} = \{0.2, 1, 3\}$ with probabilities $p_1 = 0.6$, $p_2 = 0.3$ and $p_3 = 0.1$. The optimal expected sleep duration is then $0.72\alpha^\star$.

We vary the normalized weight ϵ between 0.001 and 1.

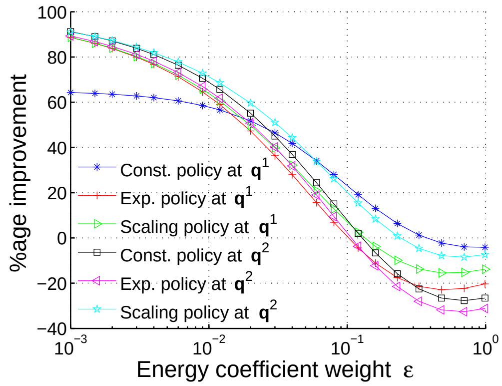


Fig. 6. Percentage improvement over standard policy for hyper-exponential τ at $\boldsymbol{\lambda} = [0.01, 2, 10]$.

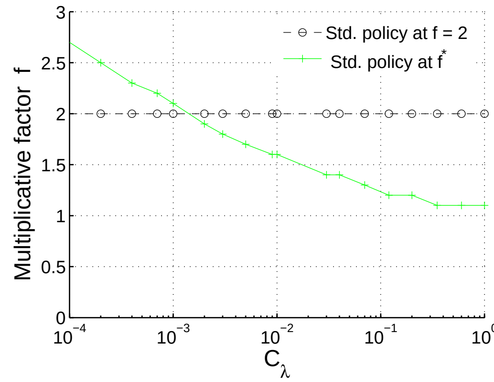


Fig. 7. $f^\star$ against standard vs. C_λ.

The optimal sleep duration (constant policy) and the optimal expected sleep duration (exponential policy) are depicted in Fig. 5(a). We observe the same thing as with Poisson arrivals (cf. Fig. 3(a)) except for the constant policy at $\boldsymbol{q}^1$ (moderate incoming traffic): the impact of ϵ on the optimal sleep size is not significant. We plan to investigate more this situation in the near future.

Figure 5(b) displays both optimal costs and the cost under the standard policy against ϵ for the two considered distributions. Here again, we observe the same trends of the curves as with Poisson arrivals (cf. Fig. 3(b)) except for the curve showing the optimal cost under the constant policy at $\boldsymbol{q}^1$ (red curve). It actually has a similar trend to that of the standard policy. It appears that for the distributions considered and $\epsilon < 0.1$, all proposed policies outperform the standard policy. However, as ϵ increases, the standard policy seems to be the best among the considered policies.

This is clearly observable in Fig. 6 which depicts the percentage improvement as a function of ϵ. Observe that the constant policy under moderate incoming traffic ($\boldsymbol{q}^1$) is outperformed by the standard policy for $\epsilon \gtrsim 0.4$, as the improvement becomes negative. We can infer from Fig. 6 that neither the constant nor the exponential and nor the scaling policies are optimal when the off time τ is hyper-exponentially distributed. It is still an open issue to idnetify the optimal policy in such situations.

We now move to the study of the standard and the multiplicative policy (cf. Sect. V-A). We want to compute the optimal multiplicative factor for a variety of disrtibutions of τ. For this purpose, we introduce a scaling factor C_λ which scales the rate vector of the hyper-exponential distribution of τ, such that the effective rate vector is given by $\boldsymbol{\lambda}_{\text{eff}} = C_\lambda \boldsymbol{\lambda}$. For illustration, we use $\boldsymbol{\lambda} = [0.2, 3, 10]$.

The optimal multiplicative factor $f^\star$ from (18) is depicted in Fig. 7 against C_λ. The horizontal line at the value 2 represents the multiplicative factor used in the standard policy. Observe that $f^\star$ approaches to 1 as C_λ increases (i.e., as the incoming traffic rate becomes higher). Recall that $f = 1$ corresponds to the situation in which successive sleep durations are of equal size. In other words, $f = 1$ is nothing but the constant policy. At the limit, as $C_\lambda \to \infty$, we expect $f^\star$ to reach the value 1, which implies that the constant policy will be a better choice than the standard policy.

VII. Concluding Remarks

We introduced an optimization framework for controlling the vacation length as well as selecting the best vacation policy. The approach can be directly applied to centrally controlled wireless devices for optimizing individual energy saving while taking into account the delays. We demonstrated important properties of several policies that can have practical interest. Also, we show the percentage of improvement of a well selected policy over a standardized policy. As future work, one can produce the derivation of conditional residual interarrival time in order to map the distribution of the idle period to exogenous arrivals. Using dynamic programming, one can show that in case of Poisson idle periods, the optimal sleep policy should be constant. Finally, the results give directions towards deriving optimal policies for unknown idle periods where the statistics should be estimated.

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APPENDIX

A. Exponentially Distributed Vacations

Proposition A.1: The cost $V_e(b)$ is a convex function having a minimum at

$$b_e^\star = \sqrt{\frac{\epsilon E_L \mathbb{E}[\tau]}{\eta}} = \sqrt{\frac{\epsilon E_L \mathbb{E}[\tau]}{\bar{\epsilon} + \epsilon P_S}}. \tag{22}$$

The minimal cost is

$$V_e(b_e^\star) = \epsilon(P_S \mathbb{E}[\tau] + E_L) + 2\sqrt{\epsilon \eta E_L \mathbb{E}[\tau]} \tag{23}$$

Proof: Let us compute the first and second derivative of the cost. We find

$$V_e'(b) = \eta - \frac{\epsilon E_L \mathbb{E}[\tau]}{b^2}$$
$$V_e''(b) = 2\frac{\epsilon E_L \mathbb{E}[\tau]}{b^3}.$$

Clearly, $V_e''(b) \geq 0$ for any positive b, hence $V_e(b)$ is a convex function. The derivative $V_e'(b)$ has a root at $b_e^\star$ as given in (22), which yields a minimum in the cost $V_e(b)$ at $b_e^\star$. Substituting the optimal $b_e^\star$ in (9) we obtain the minimal cost 23.

$\diamondsuit$

B. Equally Sized Vacations

The second strategy considered is that where vacations are constant, in other words, $B = b$. The performance is optimized by controlling the size of b. The Laplace-Stieltjes transform of B becomes $\mathcal{B}^*(s) = \exp(-sb)$, yielding the following simplified expression for the cost (the subscript stands for "constant")

$$V_c(b) = -\bar{\epsilon}\mathbb{E}[\tau] + (\epsilon E_L + \eta b) \sum_{i=1}^{n} \frac{q_i}{1 - \exp(-\lambda_i b)}. \tag{24}$$

Proposition A.2: When $n = 1$, the cost $V_c(b)$ is a convex function having a minimum at

$$b_c^\star = -\frac{1}{\lambda_1}\left(\zeta_1 + W_{-1}(-e^{-\zeta_1})\right) \tag{25}$$
$$\text{with } \zeta_1 := \frac{\lambda_1 \epsilon E_L}{\eta} + 1,$$

where $W_{-1}(-e^{-\zeta_1})$ denotes the branch of the Lambert W function[2] that is real-valued on the interval $[-\exp(-1), 0]$ and always below -1. The minimal cost is

$$V_c(b_c^\star) = -\frac{1}{\lambda}\left(\bar{\epsilon} + \eta W_{-1}(-e^{-\zeta_1})\right). \tag{26}$$

Proof: The derivative function of $V_c(b)$ is

$$V_c'(b) = \eta\left\{\frac{1 - \exp(-\lambda_1 b)(\zeta_1 + \lambda_1 b)}{(1 - \exp(-\lambda_1 b))^2}\right\}. \tag{27}$$

The extremum of $V_c(b)$, denoted $b_c^\star$, must verify $V_c'(b_c^\star) = 0$. In other words, we must have

$$\begin{aligned} 1 - \exp(-\lambda_1 b_c^\star)(\zeta_1 + \lambda_1 b_c^\star) &= 0 \\ \Leftrightarrow \quad \exp(-\zeta_1 - \lambda_1 b_c^\star)(-\zeta_1 - \lambda b_c^\star) &= -\exp(-\zeta_1). \end{aligned}$$

The last expression is of the form $y\exp(y) = x$ with $y = -\zeta_1 - \lambda_1 b_c^\star$ and $x = -\exp(-\zeta_1)$. The solution y is the Lambert W function [4], denoted W, at the point x. Hence,

$$-\zeta_1 - \lambda_1 b_c^\star = W(-\exp(-\zeta_1)).$$

Since $\zeta_1 \geq 1$, we have $-\exp(-1) \leq -\exp(-\zeta_1) < 0$. Therefore, we need $W(-\exp(-\zeta_1))$ to be real-valued in $[-\exp(-1), 0[$. Also, given that $\zeta_1 + \lambda_1 b_c^\star \geq 1$, we need $W(-\exp(-\zeta_1))$ to be always negative and smaller than -1. Both conditions are satisfied by the branch numbered -1. Hence, $-\zeta_1 - \lambda_1 b_c^\star = W_{-1}(-\exp(-\zeta_1))$ and (25) is readily found. Replacing (25) in (24) with $n = 1$, and using the relation $\exp(y) = x/y$, one can derive (26).

Now to know whether $b_c^\star$ is a maximum or a minimum, we study the second derivative function of $V_c(b)$, namely

$$V_c''(b) = \frac{\eta \lambda_1 e^{-\lambda_1 b}}{(1 - e^{-\lambda_1 b})^3}\left\{(1 + e^{-\lambda_1 b})(1 + \zeta_1 + \lambda_1 b) - 4\right\}.$$

The sign of $V_c''(b)$ depends on the value of

$$z_1(b) := (1 + \exp(-\lambda_1 b))(1 + \zeta_1 + \lambda_1 b).$$

The following can be easily derived

$$z_1'(b) = \lambda_i\left(1 - \exp(-\lambda_1 b)(\zeta_1 + \lambda_1 b)\right)$$
$$\lim_{b\to 0} z_1'(b) = -\lambda_1(1 - \zeta_1) < 0$$
$$\lim_{b\to\infty} z_1'(b) = \lambda_1 > 0$$

The derivative $z_1'(b)$ is null for $b = b_c^\star > 0$, negative for $b < b_c^\star$ and positive for $b > b_c^\star$. Hence, $z_1(b)$ decreases from $\lim_{b\to 0} z_1(b) = 2(1 + \zeta_1) > 4$ to its minimum $z_1(b_c^\star) =$

[2] The Lambert W function, satisfies $W(x)\exp(W(x)) = x$. As the equation $y\exp(y) = x$ has an infinite number of solutions y for each (non-zero) value of x, the function $W(x)$ has an infinite number of branches.

$-\frac{(W_{-1}(-e^{-\zeta_1})-1)^2}{W_{-1}(-e^{-\zeta_1})} > 4$ and then increases asymptotically to $+\infty$. We have shown that $z_1(b) > 4$ for any positive b. Therefore, $V_c''(b) > 0$ for any positive b. $V_c(b)$ is a then convex function and the extremum $b_c^\star$ is a global minimum, which concludes the proof.

$\diamondsuit$

C. Equally Sized vacation for Hyper exponential Arrival

Proposition A.3: The cost $V_c(b)$ is a convex function having a minimum in $]0, \infty[$.

Proof: The case when $n = 1$ is covered in Prop. A.2. We focus then on the case $n > 1$. For convenience, we introduce $\zeta_i = \frac{\lambda_i \epsilon E_L}{\eta} + 1$ for $i = 1, \ldots, n$. The first and second derivative functions of $V_c(b)$ are, respectively

$$V_c'(b) = \eta \sum_{i=1}^{n} q_i \left\{ \frac{1 - \exp(-\lambda_i b)(\zeta_i + \lambda_i b)}{(1 - \exp(-\lambda_i b))^2} \right\} \quad (28)$$

$$V_c''(b) = \sum_{i=1}^{n} \frac{\eta q_i \lambda_i e^{-\lambda_i b}}{(1 - e^{-\lambda_i b})^3} \left\{ (1 + e^{-\lambda_i b})(1 + \zeta_i + \lambda_i b) - 4 \right\}.$$

To study the sign of $V_c''(b)$, we need to evaluate the functions

$$z_i(b) := (1 + \exp(-\lambda_i b))(1 + \zeta_i + \lambda_i b)$$

for $i = 1, \ldots, n$. In the proof of Prop. A.2, the function $z_1(b)$ has been found to be always above 4. Similarly, the function $z_i(b)$ has a minimum at $b_i := -\frac{1}{\lambda_i}\left(\zeta_i + W_{-1}(-e^{-\zeta_i})\right)$, and $z_i(b_i)$ is always above 4, for any $i \in \{1, \ldots, n\}$. Hence, $V_c''(b) \geq 0$ for any positive b, implying that $V_c(b)$ is a convex function (the derivative $V_c'(b)$ increases with b).

We have

$$\lim_{b \to \infty} V_c'(b) = \eta = \bar{\epsilon} + \epsilon P_S > 0$$
$$\lim_{b \to 0} V_c'(b) = -\infty$$

which implies that there exists some $b_c^\star > 0$ such that $V_c'(b_c^\star) = 0$. Therefore, $V_c(b)$ has a global (strictly positive) minimum at $b_c^\star$.

$\diamondsuit$

Proposition A.4: When $n > 1$, no optimal policy can be independent of $\boldsymbol{q} = (q_1, \ldots, q_n)$.

Proof: We prove Prop. A.4 by contradiction. We assume that there exists an optimal control $\boldsymbol{\mu}^\star = (b_1^\star, b_2^\star, \ldots)$ which does not depend on $\boldsymbol{q} = (q_1, \ldots, q_n)$. Therefore, the coefficients of the probabilities $\{q_i\}_{i=1,\ldots,n}$ in (20) must all be null when $\boldsymbol{\mu} = \boldsymbol{\mu}^\star$. In other words, we must have for $i = 1, \ldots, n$ and $j \in \mathbb{N}^*$

$$\exp(-\lambda_i t_{j-1}^\star) = \sum_{k=j}^{\infty} \lambda_i \exp(-\lambda_i t_k^\star) \left(b_{k+1}^\star + \frac{\epsilon E_L}{\eta} \right).$$

Subtracting, for a given i, the expression for j from that for $j-1$, we get after simplification

$$b_{j+1}^\star = \frac{\exp(\lambda_i b_j^\star) - 1}{\lambda_i} - \frac{\epsilon E_L}{\eta} \quad (29)$$

which must hold for $j \in \mathbb{N}^*$ and for any $i \in \{1, \ldots, n\}$. Since $b_{j+1}^\star$ must be constant, it is imperative that

$$\frac{\exp(\lambda_1 b_j^\star) - 1}{x} = \ldots = \frac{\exp(\lambda_n b_j^\star) - 1}{x} \quad (30)$$

for $j \in \mathbb{N}^*$. For a given j, the above equality holds only when $b_j^\star = 0$, i.e., if there were no jth vacation. Given that (30) must hold for $j \in \mathbb{N}^*$, then all vacations need to be null. However, on the other hand, replacing $b_j^\star = 0$ in (29) yields $b_{j+1}^\star = -\epsilon E_L/\eta < 0$, which is absurd. Equations (29) and (30) contradict each other. Therefore, the starting hypothesis is wrong and the optimal control must depend on $\boldsymbol{q} = (q_1, \ldots, q_n)$.

$\diamondsuit$

The contradiction between (29) and (30) arises because $n > 1$. It is therefore expected to obtain more results when $n = 1$, which is equivalent to the case of Poisson arrival with rate λ_1. When $n = 1$, Eq. (20) becomes

$$\frac{\partial V_d(\boldsymbol{\mu})}{\partial b_j} = \eta \left[e^{-\lambda_1 t_{j-1}} - \sum_{k=j}^{\infty} \lambda_1 e^{-\lambda_1 t_k} \left(b_{k+1} + \frac{\epsilon E_L}{\eta} \right) \right].$$

The optimal vacation sizes can then be computed recursively using (29), which becomes

$$b_{j+1}^\star = \frac{\exp(\lambda_1 b_j^\star) - 1}{\lambda_1} - \frac{\epsilon E_L}{\eta}. \quad (31)$$

We can compute all vacations in terms of the first vacation $b_1^\star$, which still needs to be computed. There are actually an infinite number of solutions of $\mathbf{grad}\, V_d(\boldsymbol{\mu}^\star) = 0$, one solution for each possible value of $b_1^\star$. Not all of them correspond to minima and only one of them corresponds to a global minimum of $V_d(\boldsymbol{\mu})$. At this point, we are not able to identify the global minimum.

Observe that if we set

$$b_1^\star = -\frac{1}{\lambda_1}\left(\zeta_1 + W_{-1}(-e^{-\zeta_1})\right)$$

then (31) yields that all the $b_j^\star$'s will be equal to $b_1^\star$ (Vice-versa, letting $b_j^\star = b_{j+1}^\star$ in (31) yields (25).). In other words, the optimal constant policy given in (25) corresponds to one of the minima of $V_d(\boldsymbol{\mu})$.